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Tetrahedral meshing via maximal Poisson-disk sampling

Jianwei Guo [a], Dong-Ming Yan [a,*], Li Chen [b], Xiaopeng Zhang [a], Oliver Deussen [c,d], Peter Wonka [e]

[a] National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences, Beijing 100190, China
[b] School of Software, Tsinghua University, Beijing 100084, China
[c] University of Konstanz, Konstanz 78457, Germany
[d] Shenzhen Key Lab of Visual Computing and Visual Analytics/SIAT, Shenzhen 518055, China
[e] Visual Computing Center, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia

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ABSTRACT

In this paper, we propose a simple yet effective method to generate 3D-conforming tetrahedral meshes from closed 2-manifold surfaces. Our approach is inspired by recent work on maximal Poisson-disk sampling (MPS), which can generate well-distributed point sets in arbitrary domains. We first perform MPS on the boundary of the input domain, we then sample the interior of the domain, and we finally extract the tetrahedral mesh from the samples by using 3D Delaunay or regular triangulation for uniform or adaptive sampling, respectively. We also propose an efficient optimization strategy to protect the domain boundaries and to remove slivers to improve the meshing quality. We present various experimental results to illustrate the efficiency and the robustness of our proposed approach. We demonstrate that the performance and quality (e.g., minimal dihedral angle) of our approach are superior to current state-of-the-art optimization-based approaches.

1. Introduction

Mesh generation aims to approximate a given closed domain with simple discrete elements (e.g., triangle/quad in 2D and tetrahedron/pyramid/prism/hexahedron in 3D). It has numerous applications ranging from engineering to scientific research, e.g., simulation of mechanical parts or architecture, medical and biological data analysis, geographical science, computational fluid dynamics, and animation in computer graphics (Klingner et al., 2006; Ando et al., 2013). Here we focus on 3D tetrahedral mesh generation.

Although many robust commercial software (e.g., Ansys) and open source packages exist for mesh generation (e.g., TetGen (Si, 2015), CGALmesh (Jamin et al., 2014), DelPSC (Cheng et al., 2007), Gmsh (Geuzaine and Remacle, 2009), Qhull (Barber et al., 1996), etc.), research on tetrahedral meshing remains active because no available approach is able to fulfill all conflicting quality requirements of various applications.

There are many criteria to evaluate the quality of a tetrahedral mesh, e.g., the approximation quality, the gradation, the dihedral angle, and the radius ratio of a single tetrahedron. These criteria are typically in conflict with each other and are difficult to satisfy simultaneously. Among all the quality criteria, the dihedral angle is the most important for simulation

* Corresponding author.
E-mail addresses: jianwei.guo@nlpr.ia.ac.cn (J. Guo), yandongming@gmail.com (D.-M. Yan), chenlee@tsinghua.edu.cn (L. Chen), xpzhang@ia.ac.cn (X. Zhang), Oliver.Deussen@uni-konstanz.de (O. Deussen), pwonka@gmail.com (P. Wonka).

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because it is directly related to the condition number of the stiffness matrix. A single poorly shaped tetrahedron with near zero volume (called a *sliver*) can destroy the whole simulation (Shewchuk, 2002). Nevertheless, recent approaches still face the difficulty of completely removing slivers, even with careful treatment of the energy functions and with post-processing, such as simulated annealing or sliver exudation (Alliez et al., 2005; Tournois et al., 2009; Chen et al., 2014).

The main contribution of this paper is to propose a very simple algorithm that improves 3D tetrahedral mesh generation compared with existing algorithms that optimize complex objective functions. Our work is inspired by recent work in maximal Poisson-disk sampling (MPS) (Ebeida and Mitchell, 2012; Yan and Wonka, 2013), which generates uniformly and randomly distributed point sets while exhibiting blue-noise properties. These extracted meshes from sampled point sets have very good geometric properties, e.g., angle bounds, edge-length bounds, etc. Our approach is a combination of randomized sampling and several effective optimization strategies that improve the lower/upper bounds of dihedral angles. Tetrahedra near the boundary surface are treated carefully such that the quality of the resulting tetrahedral mesh is provably protected. Our method works better than other greedy approaches for generating samples (e.g., Delaunay insertion). It is more efficient than optimization-based algorithms while at the same time it can improve the quality of the smallest dihedral angles.

2. Related work

Delaunay mesh generation. There are three main categories of tetrahedral meshing algorithms: (i) advancing-front-based approaches (Schöberl, 1997; Choi et al., 2003; Ito et al., 2004), (ii) octree-based methods (Labelle and Shewchuk, 2007; Bronson et al., 2014; Yu et al., 2014), and (iii) Delaunay-based meshing algorithms (Cheng et al., 2012). A complete review of these algorithms is beyond the scope of this paper. We refer the reader to a survey paper (Owen, 1998) and textbooks (Bern and Plassmann, 2000; Frey and George, 2001; Edelsbrunner, 2001) (and references therein) for a more comprehensive introduction to tetrahedral meshing algorithms. In the following, we focus on Delaunay-based meshing algorithms, which are most closely related to our work.

Delaunay-based tetrahedral meshing algorithms have been shown to be the most successful in recent years (Cheng et al., 2012). From an algorithmic perspective, Delaunay meshing techniques can be further classified into two major types: Delaunay insertion/refinement-based methods (Chew, 1997; Jamin et al., 2014; Si, 2015) and optimization-based (or variational) approaches (Alliez et al., 2005; Tournois et al., 2009; Dardenne et al., 2009; Vanderzee et al., 2010; Yan et al., 2010, 2013; Chen et al., 2014).

Delaunay insertion/refinement-based approaches usually start with an initial mesh and gradually insert new vertices at the center of a circumscribed sphere of the existing tetrahedra with the worst quality. The algorithm stops once all the user-specified criteria are satisfied, e.g., smallest dihedral angle, local edge length, approximation quality of the domain boundary, and so on. The framework of TetGen (Si, 2015) uses a series of such approaches, including incremental Delaunay algorithms for inserting vertices, constrained Delaunay algorithms for inserting constraints and a Delaunay refinement algorithm for quality mesh generation. In summary, this type of meshing technique can usually provide theoretical guarantees of the angle bound, element size and approximation error. On the other hand, it is difficult to explicitly control the desired number of vertices. The Delaunay insertion method usually inserts more vertices than necessary to reach the desired quality criteria.

The optimization-based approaches start with an initial set of randomly sampled vertices, where the number of vertices is usually specified by the user. Then, different types of energy functions are proposed to characterize the desired meshing quality. A well-known optimization-based approach is *Centroidal Voronoi Tessellation* (CVT) (Du et al., 1999) mesh generation (Du and Wang, 2003; Yan et al., 2013). CVT-based algorithms optimize the compactness of the dual Voronoi cells instead of the shape of the tetrahedra, which does not suppress any poorly shaped elements. Alliez et al. (2005) generalize *Optimal Delaunay Triangulation* (ODT) (Chen, 2004) for 2D mesh smoothing to 3D tetrahedral meshing. They propose to minimize the global mesh-dependent energy to update both vertex positions and connectivity. The work of Tournois et al. (2009), called natural ODT (NODT), further extends the ODT energy to the domain boundaries. This extension ensures consistency of the energy function on the boundary and inside the domain. The number of badly shaped elements is reduced near the domain boundary. Recent work of Chen (2004) proposes an extended formulation for ODT that is able to generate graded meshes efficiently using quasi-Newton solvers. This type of algorithm generates higher-quality meshes in practice, but they are more time consuming and post-processing for mesh improvement is necessary to eliminate badly shaped elements (Cheng et al., 2000; Klingner and Shewchuk, 2007). In addition, Jiao et al. (2011) present a variational smoothing approach for optimizing surface and volume meshes simultaneously. They define energy functions based on conformal and isometric mappings between actual elements with ideal reference elements and develop a simple algorithm to minimize the energies.

From a boundary approximation perspective, Delaunay meshing can be classified into constrained meshing or conformal meshing. Our approach falls into the latter category, i.e., instead of interpolating the boundary, we approximate the boundary by resampling, which allows for higher tetrahedron quality at the boundary, especially when the quality of the input surface is very poor. Here, we present a simple method based on randomized sampling that is able to generate high-quality meshes that are comparable with state-of-the-art techniques, while being more efficient than variational approaches.