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Convergence of barycentric coordinates to barycentric kernels

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ABSTRACT

We investigate the close correspondence between barycentric coordinates and barycentric kernels from the point of view of the limit process when finer and finer polygons converge to a smooth convex domain. We show that any barycentric kernel is the limit of a set of barycentric coordinates and prove that the convergence rate is quadratic. Our convergence analysis extends naturally to barycentric interpolants and mappings induced by barycentric coordinates and kernels. We verify our theoretical convergence results numerically on several examples.

1. Introduction

Generalised barycentric coordinates have become a key tool in many applications in geometric modelling and computer graphics, ranging from shading, deformation, and animation to colour and shape blending. Wachspress coordinates (Wachspress, 1975; Meyer et al., 2002) and mean value coordinates (Floater, 2003; Hormann and Floater, 2006) belong to the most popular generalised barycentric coordinates for polygons in the plane; see also Floater (2014). These two examples of coordinates fall into the family of three-point coordinates (Floater et al., 2006). Other barycentric coordinates for polygons that do not belong to the three-point family exist, including complex coordinates (Weber et al., 2009) and local barycentric coordinates (Zhang et al., 2014). Various applications of barycentric coordinates were discussed and summarised in Gillette et al. (2012), Rustamov (2010), Schneider et al. (2013), Hormann (2014).

In certain scenarios such as transfinite interpolation (Ju et al., 2005; Warren et al., 2007; Dyken and Floater, 2009), it is preferable to deal with smooth domains rather than polygons. Considering denser and denser polygons inscribed into a smooth domain leads to a (convergent) series of barycentric coordinates. This limit process was recently investigated in the case of Wachspress coordinates in Kosinka and Bartoň (2015). It was shown that in the limit, as the number of vertices of the polygon approaches infinity, one obtains the Wachspress kernel (Warren et al., 2007) introduced by Warren et al., and that the convergence rate is quadratic. Barycentric kernels constitute a natural generalisation of barycentric coordinates from polygons to smooth domains and their applications include deformation and transfinite interpolation (Warren et al., 2007). For more details on generalised barycentric coordinates and kernels, consult the recent survey (Floater, 2015).

In this work, we investigate the same type of convergence for the complete family of barycentric coordinates (Floater et al., 2006), which includes Wachspress and mean value coordinates as special cases. Building on the approach taken in Kosinka and Bartoň (2015), we show that any set of barycentric coordinates converges quadratically to a well-defined limit given by the corresponding barycentric kernel (Belyaev, 2006; Schaefer et al., 2007).

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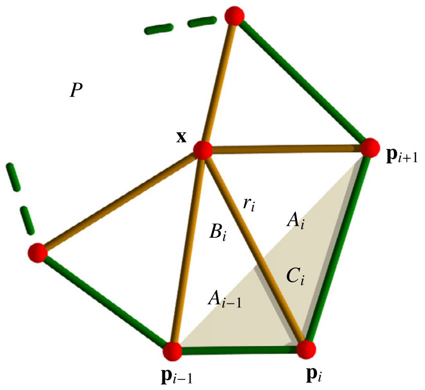

Fig. 1. Distances and signed triangle areas in the definition of three-point coordinates; see (7). The i-th weight w_i of $\mathbf{x}$ in a polygon P depends only on three consecutive vertices $\mathbf{p}_{i-1}$, $\mathbf{p}_i$, and $\mathbf{p}_{i+1}$.

After recalling some basic concepts and introducing our notation for barycentric coordinates and kernels (Section 2), we study the convergence of barycentric coordinates for a sequence of convex polygons converging to a smooth convex domain (Section 3). Our convergence analysis carries over to interpolants and mappings based on barycentric coordinates and kernels (Section 4). Numerical examples based on our theoretical convergence analysis are presented in Section 5. We conclude the paper by discussing the results and their relevance in Section 6.

2. Barycentric coordinates and kernels

The classical barycentric coordinates for triangles are unique, but there exist various generalised barycentric coordinates for planar polygons with more than three vertices. Let the vertices of a strictly convex polygon P be $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, ordered anticlockwise. (Generalised) barycentric coordinates $\lambda_i : \bar{P} \to \mathbb{R}$, $i = 1, \ldots, n$ for P are given by the following three properties:

(P1) Non-negativity

$$\lambda_i(\mathbf{x}) \geq 0, \quad i = 1, \ldots, n; \tag{1}$$

(P2) Partition of unity

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1; \tag{2}$$

(P3) Barycentric property

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{p}_i = \mathbf{x}; \tag{3}$$

for all $\mathbf{x} \in \bar{P}$, where $\bar{P}$ denotes the closure of P, which is itself understood as an open set.

Several other properties follow from (P1)–(P3), including the Lagrange property, i.e., $\lambda_i(\mathbf{p}_j) = \delta_{i,j}$, and the interpolation property of piece-wise linear functions defined on ∂P, the boundary of P, for continuous coordinates (Floater et al., 2006).

For later use, we define the three signed triangle areas

$$A_i(\mathbf{x}) = \mathcal{A}(\mathbf{x}, \mathbf{p}_i, \mathbf{p}_{i+1}), \; B_i(\mathbf{x}) = \mathcal{A}(\mathbf{x}, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}), \; C_i = \mathcal{A}(\mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}), \tag{4}$$

and the distance $r_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{p}_i\|$; see Fig. 1. The indices of A_i, B_i, $\mathbf{p}_i$ and so on are treated cyclically with respect to n.

Barycentric coordinates $\lambda_i(\mathbf{x})$ are usually defined in terms of weights $w_i(\mathbf{x})$ via

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{W(\mathbf{x})}, \quad W(\mathbf{x}) = \sum_{j=1}^{n} w_j(\mathbf{x}); \quad \mathbf{x} \in \bar{P}. \tag{5}$$

In Floater et al. (2006, Corollary 3), it was shown that any set of weights (also called homogeneous coordinates) is of the form

$$w_i(\mathbf{x}) = \frac{c_{i+1}(\mathbf{x})A_{i-1} - c_i(\mathbf{x})B_i + c_{i-1}(\mathbf{x})A_i}{A_{i-1}A_i} \tag{6}$$

for some set of real bivariate functions $c_i(\mathbf{x})$ defined on P.