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Uniform approximation of a circle by a parametric polynomial curve ☆

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ABSTRACT

In the paper two new approaches for construction of parametric polynomial approximants of a unit circle are presented. The obtained approximants have better approximation properties than those given by other methods, i.e., smaller radial error, symmetry, and exponential error decay.

1. Introduction

A circle arc is a basic object in CAGD and in many applications. Conics are the oldest curves, and are used in architecture, robotics, and in many other fields. The unit circle has a nice parameterization

$$\boldsymbol{c}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi).$$

In CAGD, parametric polynomial and rational curves and splines are fundamental objects. A circle arc does not have a parametric polynomial representation, however, it can be represented by using a quadratic rational Bézier curve. The whole circle can thus be represented by a quadratic rational spline, e.g. as a NURBS. A natural question is, whether it is possible to obtain a good parametric polynomial approximation of a circular arc.

A lot of papers study good approximation of circular segments with the radial error as the parametric distance. Quadratic Bézier approximants are considered in Mørken (1991), and their generalizations to the cubic case can be found in Dokken et al. (1990) and Goldapp (1991). The quartic case is systematically studied in Ahn and Kim (1997), Kim and Ahn (2007) and Hur and Kim (2011), and quintic Bézier approximants are derived in Fang (1998, 1999). Recently, quartic G^1 approximants were analyzed in Kovač and Žagar (2014).

General results on Hermite type approximation of conic sections by parametric polynomial curves of odd degree are given in Floater (1995, 1997). The results hold true only asymptotically, i.e., for small segments of a particular conic section. Hermite approximation of ellipse segments by cubic parametric Bézier curves is studied in Dokken (2003) and also in Dokken (1997).

In recent years some quite surprisingly good approximations of the whole circle were obtained. An approach based on Taylor approximation was improved by idea of geometric interpolation and a construction of polynomial approximants for

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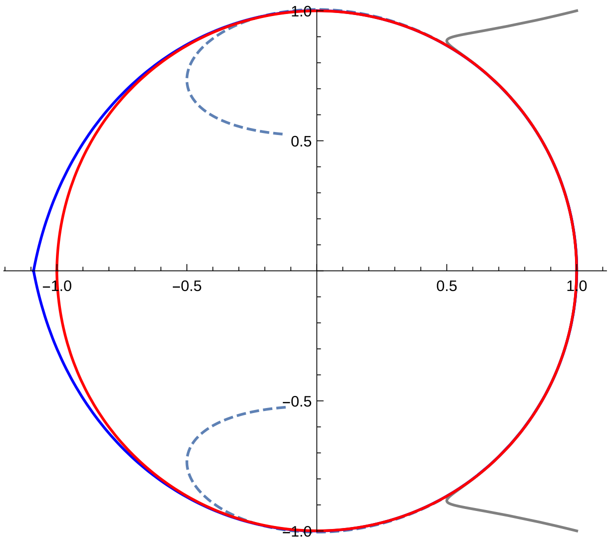

Fig. 1. Quintic parametric polynomial approximant from Lyche and Mørken (1994) (gray), parametric approximant of the same degree given in Jaklič et al. (2013) (blue), quintic Taylor approximant (blue dashed), and the new quintic approximant (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

all odd degrees was obtained in Lyche and Mørken (1994). The construction that covered also even degrees was presented in Jaklič et al. (2007a). By looking at the problem from a different perspective, it turned out that the obtained construction was just one of several solutions of a nonlinear problem, and that there exist better solutions. In Jaklič et al. (2013), the best such solution was presented, which gives a good approximation of a conic section. It has many nice properties, it is symmetric, shape preserving, it gives a high order approximation of the whole circle $\boldsymbol{c}$, and for higher degrees it circles the circle several times before it deviates from $\boldsymbol{c}$. Furthermore, it is given in a closed form.

In this paper, a novel parametric polynomial approximant for the whole circle is presented, that gives a better approximation. It has many desired properties, such as symmetry and high order approximation. Another approach is presented that yields even (slightly) better results. Since solving of nonlinear equations and systems are involved, unfortunately the solutions could not be given exactly.

As a motivation, consider Fig. 1. Here a parametric quintic polynomial approximant of the unit circle, given in Lyche and Mørken (1994), Jaklič et al. (2007a) as

$$\begin{pmatrix} 1-2t^2+2t^4 \\ 2t-2t^3+t^5 \end{pmatrix}$$

is shown together with the quintic approximant from Jaklič et al. (2013) (with radial error 0.08999)

$$\begin{pmatrix} 1-(3+\sqrt{5})t^2+(1+\sqrt{5})t^4 \\ (1+\sqrt{5})t-(3+\sqrt{5})t^3+t^5 \end{pmatrix},$$

and the quintic Taylor approximant, together with the new quintic approximant

$$\begin{pmatrix} 0.99947004-3.87624490\,t^2+1.87807588\,t^4 \\ 2.79286220\,t-3.43427162\,t^3+0.69240320\,t^5 \end{pmatrix},$$

which is visually indistinguishable from the unit circle (the radial error is 0.00052). Note that also the new quartic approximant gives a better result than the quintic approximant from Jaklič et al. (2013) with radial error 0.0109 (see Fig. 2).

Our goal is to construct parametric polynomials x_n and y_n of degree $\leq n$, which yield a good approximation of the whole unit circle

$$\cos^2(t)+\sin^2(t)=1.$$

Let us consider the expression

$$x_n^2(t)+y_n^2(t)=1+\epsilon(t). \tag{1}$$

Here, ϵ is a polynomial of degree $\leq 2n$. Since the circle does not have a parametric polynomial parameterization, every polynomial approximation (x_n, y_n) yields a deviation ϵ from the unit circle with the implicit equation

$$x^2+y^2=1.$$