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Characterization of bivariate hierarchical quartic box splines on a three-directional grid ☆

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ABSTRACT

We consider the adaptive refinement of bivariate quartic C^2-smooth box spline spaces on the three-directional (type-I) grid G. The polynomial segments of these box splines belong to a certain subspace of the space of quartic polynomials, which will be called the space of special quartics. Given a bounded domain $\Omega \subset \mathbb{R}^2$ and finite sequence $(G^\ell)_{\ell=0,\ldots,N}$ of dyadically refined grids, we obtain a hierarchical grid by selecting mutually disjoint cells from all levels such that their union covers the entire domain. Using a suitable selection procedure allows to define a basis spanning the hierarchical box spline space. The paper derives a characterization of this space. Under certain mild assumptions on the hierarchical grid, the hierarchical spline space is shown to contain all C^2-smooth functions whose restrictions to the cells of the hierarchical grid are special quartic polynomials. Thus, in this case we can give an affirmative answer to the completeness questions for the hierarchical box spline basis.

1. Introduction

Box splines and the functions contained in the spaces spanned by them form a very useful class of piecewise polynomial functions on regular grids. They possess a number of useful properties that make them well-suited for applications. It has been shown that box splines have small support (a few cells of the underlying grid), they are non-negative, they form a partition of unity, and that box splines are refinable, i.e., the box spline spaces on refined grids are nested (Böhm, 1983; de Boor and Ron, 2008). Monographs and survey articles about box splines include (de Boor et al., 1993; de Boor and Ron, 2008; Chui, 1988; Prautzsch and Boehm, 2002). Tensor-product B-splines are special instances of box splines also.

From the rich literature on box splines, we mention a few representative publications on three specific topics. Firstly, a substantial number of results on the *approximation power* of box splines is described in the literature, e.g. (Lyche et al., 2008; Ron and Sivakumar, 1993). Secondly, several publications discuss techniques for the *efficient manipulation* of box spline bases. A general stable evaluation algorithm is devised in Kobbelt (1997). In Kim and Peters (2009) the problem of efficient evaluation of box splines is addressed by making use of the local Bernstein representation of basis functions on each triangle. Also, *numerical integration* schemes, which are important for applications, based on quasi-interpolation have been considered in Lamberti (2009), Dagnino et al. (2013).

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Using *hierarchical splines* is a well-established approach to adaptive refinement in geometric modeling (Forsey and Bartels, 1988) and numerical analysis (Mustahsan, 2011; Schillinger et al., 2012; Vuong et al., 2011). Kraft (1998) introduced a basis for hierarchical tensor-product spline spaces using a selection mechanism for B-splines. More recently, a slight modification of this approach was shown to provide a basis with better properties, such as the partition of unity property, strong stability and full approximation power (Giannelli et al., 2012, 2014; Speleers and Manni, 2015). The hierarchical approach has been extended to Powell–Sabin splines, Zwart–Powell elements and B-spline-type basis functions for cubic splines on regular grids (Kang et al., 2014; Speleers et al., 2009; Zore and Jüttler, 2014).

Computations of the dimensions of spline spaces on partitions of a domain and constructions of suitable bases for such spaces are classical problems considered in the theory of multivariate splines (Wang, 2001; Zeng et al., 2015). The *completeness question* (i.e., how to construct a basis spanning the entire spline space on a partition of a domain) led to the introduction of polynomial splines over hierarchical T-meshes (PHT-splines) (Deng et al., 2006, 2008; Li et al., 2010). Bivariate splines over T-meshes are considered in Schumaker and Wang (2012). A graphical summary of the related literature has been given in Mokriš and Jüttler (2014, Fig. 1). Several publications have addressed the completeness questions for hierarchical spline spaces generated by tensor-product B-splines (Berdinsky et al., 2014; Giannelli and Jüttler, 2013; Mokriš et al., 2014). Given a hierarchical spline space, these publications derive sufficient conditions which guarantee that the associated basis (obtained by Kraft's selection mechanism) spans the entire spline space on the partition of the domain which is determined by the hierarchical space.

The paper explores the hierarchical spline spaces generated by C^2-smooth quartic box splines on nested type-I triangulations of a bounded domain $\Omega \subset \mathbb{R}^2$, cf. Fig. 1. These functions form the mathematical basis of Loop's subdivision scheme and are therefore used to construct the regular parts of the corresponding subdivision surfaces, cf. Loop (1987), Stollnitz et al. (1996). Moreover, it is known that any C^2-smooth piecewise polynomial function of degree 4 on a type-I triangulation of $\mathbb{R}^2$ can be represented as a linear combination of box splines plus three (globally supported) truncated power (piecewise polynomial) functions of degree 4. However, the box splines suffice for representing all *locally supported* C^2-spline in the space, cf. Chui and Wang (1984). On each cell of the triangulation, the space generated by the box splines spans a 12-dimensional subspace of quartic bivariate polynomials. This subspace, which will be called the space of *special quartics*, is known to contain the cubic polynomials. The current manuscript follows the approach presented in Mokriš et al. (2014) in order to establish the completeness of hierarchical C^2-smooth quartic box splines with respect to special quartics.

Using a suitable selection procedure, which generalizes the hierarchical B-spline basis introduced by Kraft (1997) to quartic C^2-smooth box splines, we define a basis spanning a hierarchical box spline space. We prove that the elements of the space can equivalently be characterized as the C^2-smooth functions whose restrictions to the cells of the hierarchical grid (which consists of mutually disjoint cells from different levels covering the entire domain) are special quartic polynomials.

The remainder of this paper consists of five sections and an appendix. The next section recalls existing results concerning bivariate spline spaces on regular grids and C^2-smooth quartic box splines. It also derives a characterization of C^2-contacts between segments of special quartic polynomials by their box spline coefficients. Section 3 is devoted to the spaces of C^2-smooth functions whose segments are special quartic polynomials on a multi-cell domain, where all cells belong to the same level. For certain multi-cell domains, which are said to be admissible, these spaces are spanned by the associated box splines. These domains are characterized by an offset condition in Section 4. Based on these results, the fifth section discusses the completeness of the hierarchical box spline basis on the associated hierarchical grids. Finally we conclude the paper. The Appendix A proves that the spaces spanned by the special quartic polynomials on each cell are simply restrictions of a globally defined subspace of the space of quartic polynomials.

2. Preliminaries

We recall existing results concerning C^2-smooth quartic splines on regular type-I triangulations and characterize C^2-smooth contacts between special quartic polynomials on adjacent triangular cells.

2.1. Bivariate splines on regular grids

We consider bivariate splines on a three-directional grid in the plane $\mathbb{R}^2$, see Fig. 1. Let us denote by $\mathcal{P}_d$ the linear space of polynomials in $\mathbb{R}[x, y]$ of degree less than $d+1$.

Furthermore, we consider a partition G_Ω of a polygonal domain $\Omega \subset \mathbb{R}^2$ into mutually disjoint cells, where each cell is an open set and the closure of the union of all cells equals Ω. In addition we choose a finite-dimensional linear space T of functions on $\mathbb{R}^2$.

Definition 1. We define $S^r(G_\Omega, T)$ to be the space of C^r-smooth functions s on Ω with the property that their restrictions (denoted by the vertical bar $s|_\triangle$) to any cell $\triangle \in G_\Omega$ yields a function which is obtained by restricting a function in T to the cell, i.e.,

$$S^r(G_\Omega, T) = \{s \in C^r(\Omega) : s|_\triangle \in T|_\triangle \text{for all cells } \triangle \in G_\Omega\}, \tag{1}$$

where the linear space $T|_\triangle$ contains the restrictions of the functions in T to the cell $\triangle$.