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Rational swept surface constructions based on differential and integral sweep curve properties ☆

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ABSTRACT

A *swept surface* is generated from a profile curve and a sweep curve by employing the latter to define a continuous family of transformations of the former. By using polynomial or rational curves, and specifying the homogeneous coordinates of the swept surface as bilinear forms in the profile and sweep curve homogeneous coordinates, the outcome is guaranteed to be a rational surface compatible with the prevailing data types of CAD systems. However, this approach does not accommodate many geometrically intuitive sweep operations based on differential or integral properties of the sweep curve — such as the parametric speed, tangent, normal, curvature, arc length, and offset curves — since they do not ordinarily have a rational dependence on the curve parameter. The use of Pythagorean-hodograph (PH) sweep curves surmounts this limitation, and thus makes possible a much richer spectrum of rational swept surface types. A number of representative examples are used to illustrate the diversity of these novel swept surface forms — including the oriented-translation sweep, offset-translation sweep, generalized conical sweep, and oriented-involute sweep. In many cases of practical interest, these forms also have rational offset surfaces. Considerations related to the automated CNC machining of these surfaces, using only their high-level procedural definitions, are also briefly discussed.

1. Introduction

Sweep operations provide an intuitive approach to constructing surfaces from two parametric curves, a *profile curve* $\mathbf{p}(u)$ and *sweep curve* $\mathbf{s}(v)$. The points of the sweep curve determine a family of transformations acting on the profile curve, such that the continuum of its transformed instances generates a *swept surface* $\mathbf{R}(u, v)$. In addition to the profile and sweep curves, a swept surface construction requires a precise specification of the nature of the geometrical transformation associated with each point of the sweep curve.

Two familiar cases are the *surface of extrusion* and *surface of revolution*. In the former case, the sweep curve is a line segment, whose points specify a family of translations of the profile curve. In the latter case, the sweep curve is a circular arc, whose points specify a family of rotations of the profile curve about an axis orthogonal to the plane of the circle. Ruled surfaces, quadrics, and cyclides are further elementary surfaces that can be defined by intuitive sweep operations. These are just the simplest examples of the diverse surface types that can be constructed using generalized sweep operations.

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Although many current commercial CAD systems incorporate some swept surface capability, it is often of limited scope or incurs approximation of the precise surface geometry. The universal reliance of CAD systems on rational parametric surface representations limits the swept surface types that can be exactly generated, and necessitates the use of data-intensive approximations for those that cannot. The intent of the present study is to illustrate how the family of exact rational swept surface constructions can be greatly expanded by employing *Pythagorean-hodograph* (PH) *curves* (Farouki, 2008) as sweep curves. The PH curves possess rational differential and integral properties — parametric speed, arc length, tangent and normal, curvature, and offset curves — that facilitate the construction of a rich variety of rational swept surfaces in which the geometrical transformations of the profile curve are explicitly dependent on these properties of the sweep curve, not just its point coordinates.

Perhaps the earliest systematic approach to swept surface constructions was proposed in the late 1970s by Hinds and Kuan in two little-known papers (Hinds and Kuan, 1978, 1979) (see also Section 13.8 of Farouki, 2008). In their methodology, an elegant matrix algebra is employed to directly construct the swept surface $\mathbf{R}(u,v)$ from the profile and sweep curves $\mathbf{p}(u)$ and $\mathbf{s}(v)$. Assuming that they are both rational cubic curves, they can be represented by 4×4 matrices in which one index is associated with a homogeneous coordinate component, and the other with the cubic basis functions. Similarly, $\mathbf{R}(u,v)$ can be represented as a rational bicubic surface through a $4\times 4\times 4$ matrix, in which one index is associated with a homogeneous coordinate component, and the others with cubic basis functions in u and v. The $4\times 4\times 4$ matrix that defines $\mathbf{R}(u,v)$ is generated by contracting the 4×4 matrices defining $\mathbf{p}(u)$ and $\mathbf{s}(v)$ with a $4\times 4\times 4$ *selector matrix*, that serves to specify the nature of the geometrical transformations of the profile curve $\mathbf{p}(u)$ associated with each point of the sweep curve $\mathbf{s}(v)$.

The essence of this approach lies in judicious choice of the selector matrix elements, so as to implement the desired sweep operation. The coordinates of the sweep curve points may be used to define translations, scalings, rotations, and perspectivities of the profile curve — but only in certain combinations, determined by linear dependence of the homogeneous coordinates of $\mathbf{R}(u,v)$ on those of $\mathbf{p}(u)$ and $\mathbf{s}(v)$. For example, if the profile and sweep curves are planar, it is not possible to define a translational sweep of $\mathbf{p}(u)$ along $\mathbf{s}(v)$, such that $\mathbf{p}(u)$ always lies in the normal plane of $\mathbf{s}(v)$, since the orientation of this plane does not ordinarily have a rational dependence on v. However, if $\mathbf{s}(v)$ is a PH curve, it has a rational unit normal vector, so a rational surface $\mathbf{R}(u,v)$ defined by the above sweep operation becomes possible.

The approach proposed herein may be considered a generalization of the rational swept surface constructions introduced by Hinds and Kuan (1978, 1979) to accommodate geometrical transformations dependent on certain differential and integral properties of a Pythagorean-hodograph sweep curve $\mathbf{s}(v)$. This opens up a much richer class of swept surface constructions amenable to exact representation within the prevailing geometry data types of CAD systems. As in Hinds and Kuan (1978, 1979) a key motivation is the ability to fabricate such surfaces on CNC machines, using only a high-level procedural definition of the surface. In this context, the ability to efficiently compute certain surface properties, such as the unit surface normal vector, is essential. However, the focus of this paper is on the construction of swept surfaces and computation of their geometrical properties. Since the CNC machining application incurs additional technical considerations, it will be deferred to a subsequent study.

The plan for the remainder of this paper is as follows. Section 2 reviews some preparatory material on the homogeneous-coordinate representation of rational curves and surfaces, the construction and properties of planar PH curves, and some basic principles for the construction and machining of swept surfaces. Section 3 describes the *scaled-rotation sweep*, a representative case of the "traditional" swept surface types defined by bilinear forms in the profile and sweep curve homogeneous coordinates, and some intrinsic limitations of these forms are identified. The generalization to sweep operations dependent on differential or integral properties of a (PH) sweep curve yields a remarkable variety of new possibilities. The goal of this paper is to introduce these novel surface types through illustrative examples of practical interest, rather than to attempt an exhaustive categorization. Consequently, Sections 4–7 present basic definitions, constructions, and examples for several important cases — the oriented-translation sweep, offset-translation sweep, oriented-involute sweep, and generalized conical sweep. For simplicity, these cases all employ planar sweep curves, but Section 8 briefly describes a type of swept surface that addresses the additional orientational freedom associated with a spatial sweep curve: the *rotation-minimizing* sweep. Finally, Section 9 summarizes and assesses the key contributions of this study, and identifies some promising directions for further extending and applying them.

2. Rational swept surfaces

The term *swept surface* (or *swept volume*) is often used to refer to the region defined as the union of the instances of a rigid body that executes a general spatial motion — such surfaces are of great importance in modeling material removal in CNC machining, collision avoidance in robotics, and related fields (Blackmore and Leu, 1992; Chiou and Lee, 2002; Chung et al., 1998; Crossman and Yoon, 2001; Li et al., 2012; Martin and Stephenson, 1990; Wang et al., 2000; Wang and Wang, 1986). A closely related concept is the *generalized cylinder* (or *cone*) that has seen widespread application as a basic shape primitive in computer vision and biomedical applications (Behrens et al., 2003; Delibasis et al., 2013; Gansca et al., 2002; Nevatia and Binford, 1977; Raghupathi et al., 2004; Semwal and Hallauer, 1994; Ulupinar and Nevatia, 1995).

In the present context, we are interested in the process of using two curves to intuitively *design* a smooth surface, by imposing a continuous family of transformations on an "initial" (profile) curve, parameterized by the points of the other (sweep) curve. The emphasis is on achieving greater flexibility in the types of the allowed transformations, subject to the