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The inverse of a rational bilinear mapping ☆

Michael S. Floater

Department of Mathematics, University of Oslo, PO Box 1053, Blindern, 0316 Oslo, Norway

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ABSTRACT

We study the problem of inverting rational bilinear mappings, which leads to a one-parameter family of generalized barycentric coordinates for quadrilaterals, including Wachspress coordinates as a special case.

1. Introduction

In a recent paper, Sederberg and Zheng (2015) studied rational bilinear mappings from the unit square to a convex quadrilateral, and their inverses. Such mappings have played an important role in computer graphics and geometric design (Wolberg, 1990). If $P \subset \mathbb{R}^2$ is the quadrilateral, with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ in $\mathbb{R}^2$, as in Fig. 1, and weights $w_1, w_2, w_3, w_4 > 0$ are chosen, the mapping

$$\mathbf{r}(s,t) := \frac{(1-s)(1-t)w_1\mathbf{v}_1 + s(1-t)w_2\mathbf{v}_2 + stw_3\mathbf{v}_3 + (1-s)tw_4\mathbf{v}_4}{(1-s)(1-t)w_1 + s(1-t)w_2 + stw_3 + (1-s)tw_4},$$

is a bijection $\mathbf{r} : [0,1] \times [0,1] \to P$. This can be seen from the fact that for a fixed s in $[0,1]$, $\mathbf{r}$ is a line segment connecting a point on the edge $[\mathbf{v}_1, \mathbf{v}_2]$ to a point on $[\mathbf{v}_4, \mathbf{v}_3]$, and as s increases from 0 to 1, the convexity of P ensures that these line segments cover every point of P once only. Thus $\mathbf{r}$ has an inverse and for any point $\mathbf{x} \in P$, we can solve

$$\mathbf{r}(s,t) = \mathbf{x}, \tag{1}$$

uniquely for s and t in $[0,1]$. Sederberg and Zheng (2015) derived a condition on the weights that makes the inversion particularly simple. For $i = 1, 2, 3, 4$, let C_i denote the triangle area,

$$C_i = A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}),$$

where vertices are indexed cyclically, $\mathbf{v}_{i+4} = \mathbf{v}_i$, $i \in \mathbb{Z}$. They showed that if

$$\frac{w_1 w_3}{w_2 w_4} = \frac{C_1 C_3}{C_2 C_4}, \tag{2}$$

then s and t are rational functions of $\mathbf{x}$.

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E-mail address: michaelf@math.uio.no.

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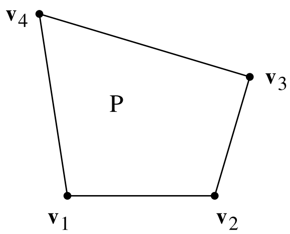

Fig. 1. Convex quadrilateral.

In this short note we firstly point out that under condition (2), s and t can be obtained from the area form of Wachspress' rational coordinates for convex polygons. Secondly, we derive a solution for s and t for arbitrary weights w_i. From this general solution we obtain generalized barycentric coordinates for P that do not seem to have been studied previously. They form a one-parameter family of coordinates, including Wachspress coordinates as a special case.

2. Wachspress coordinates

Wachspress developed rational barycentric coordinates for convex polygons in (Wachspress, 1975). The quadrilateral case has been studied in (Gout, 1979) and (Dahmen et al., 2000). Meyer et al. (2002) found a formula for the coordinates in terms of triangle areas, which for the quadrilateral P means that $\mathbf{x} \in P$ can be expressed as

$$\mathbf{x} = \frac{C_1 A_2 A_3 \mathbf{v}_1 + C_2 A_3 A_4 \mathbf{v}_2 + C_3 A_4 A_1 \mathbf{v}_3 + C_4 A_1 A_2 \mathbf{v}_4}{C_1 A_2 A_3 + C_2 A_3 A_4 + C_3 A_4 A_1 + C_4 A_1 A_2}, \tag{3}$$

where, in addition to the triangle areas C_i, the A_i are also triangle areas,

$$A_i = A_i(\mathbf{x}) = A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

An example of weights w_i that satisfy condition (2) is $w_i = C_i$. In this case, we see that by dividing numerator and denominator of (3) by $(A_4 + A_2)(A_1 + A_3)$ the solution to (1) is

$$s = \frac{A_4}{A_4 + A_2}, \qquad t = \frac{A_1}{A_1 + A_3}.$$

For general weights satisfying (2), we can first multiply the numerator and denominator of (3) by w_1/C_1 to express it as

$$\mathbf{x} = \frac{A_2 A_3 w_1 \mathbf{v}_1 + \rho_1 A_3 A_4 w_2 \mathbf{v}_2 + \rho_1 \rho_2 A_4 A_1 w_3 \mathbf{v}_3 + \rho_2 A_1 A_2 w_4 \mathbf{v}_4}{A_2 A_3 w_1 + \rho_1 A_3 A_4 w_2 + \rho_1 \rho_2 A_4 A_1 w_3 + \rho_2 A_1 A_2 w_4},$$

where

$$\rho_1 := \frac{w_1 C_2}{C_1 w_2}, \qquad \rho_2 := \frac{w_1 C_4}{C_1 w_4},$$

and then we see that the solution to (1) under condition (2) is

$$s = \frac{\rho_1 A_4}{\rho_1 A_4 + A_2}, \qquad t = \frac{\rho_2 A_1}{\rho_2 A_1 + A_3}.$$

3. General solution

Consider now solving the inversion for arbitrary weights w_i. We can adapt the formula derived recently in (Floater, 2015) for the inverse of the bilinear mapping

$$\mathbf{p}(s,t) := (1-s)(1-t)\mathbf{v}_1 + s(1-t)\mathbf{v}_2 + st\mathbf{v}_3 + (1-s)t\mathbf{v}_4. \tag{4}$$

For any point $\mathbf{x} \in P$ the solution (s,t) to

$$\mathbf{p}(s,t) = \mathbf{x} \tag{5}$$

can be expressed as follows. For $i = 1, 2, 3, 4$, define the vectors $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$. Then (5) can be expressed as

$$(1-s)(1-t)\mathbf{d}_1 + s(1-t)\mathbf{d}_2 + st\mathbf{d}_3 + (1-s)t\mathbf{d}_4 = 0.$$

Then, using the fact that