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Symmetry detection of rational space curves from their curvature and torsion [☆]

Juan Gerardo Alcázar [a,1], Carlos Hermoso [a], Georg Muntingh [b,*]

[a] *Departamento de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain*
[b] *SINTEF ICT, PO Box 124, Blindern, 0314 Oslo, Norway*

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ABSTRACT

We present a novel, deterministic, and efficient method to detect whether a given rational space curve is symmetric. By using well-known differential invariants of space curves, namely the curvature and torsion, the method is significantly faster, simpler, and more general than an earlier method addressing a similar problem (Alcázar et al., 2014b). To support this claim, we present an analysis of the arithmetic complexity of the algorithm and timings from an implementation in `Sage`.

1. Introduction

The problem of detecting the symmetries of curves and surfaces has attracted the attention of many researchers throughout the years, because of the interest from fields like Pattern Recognition (Boutin, 2000; Calabi et al., 1998; Huang and Cohen, 1996; Lebmeir and Richter-Gebert, 2008; Lebmeir, 2009; Suk and Flusser, 1993, 2005; Tarel and Cooper, 2000; Taubin and Cooper, 1992; Weiss, 1993), Computer Graphics (Berner et al., 2008; Bokeloh et al., 2009; Lipman et al., 2010; Martinet et al., 2006; Mitra et al., 2006; Podolak et al., 2006; Schnabel et al., 2008; Simari et al., 2006), and Computer Vision (Alt et al., 1988; Brass and Knauer, 2004; Jiang et al., 1996; Li et al., 2008, 2010; Loy and Eklundh, 2006; Tate and Jared, 2003; Sun and Sherrah, 1997). The introduction in Alcázar et al. (2014b) contains an extensive account of the variety of approaches used in the above references.

A common characteristic in most of these papers is that the methods focus on computing *approximate* symmetries more than exact symmetries, which is perfectly reasonable in many applications, where curves and surfaces often serve as merely approximate representations of a more complex shape. Some exceptions appear here: If the object to be considered is discrete (e.g. a polyhedron), or is described by a discrete object, like for instance a control polygon or a control polyhedron, then the symmetries can be determined exactly (Alt et al., 1988; Brass and Knauer, 2004; Jiang et al., 1996; Li et al., 2008). Examples of the second class are Bézier curves and tensor product surfaces. Furthermore, in these cases the symmetries of the curve or surface follow from those of the underlying discrete object. Another exception appears in Lebmeir and Richter-Gebert (2008), where the authors provide a deterministic method to detect rotation symmetry of an

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* Corresponding author.
E-mail addresses: juange.alcazar@uah.es (J.G. Alcázar), carlos.hermoso@uah.es (C. Hermoso), georgmu@math.uio.no (G. Muntingh).
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implicitly defined algebraic plane curve and to find the exact rotation angle and rotation center. The method uses a complex representation of the curve and is generalized in Lebmeir (2009) to detect mirror symmetry as well.

Rational curves are frequently used in Computer Aided Geometric Design and are the building blocks of NURBS curves. Compared to implicit curves, rational parametric curves are easier to manipulate and visualize. Space curves appear in a natural way when intersecting two surfaces, and they play an important role when dealing with special types of surfaces, often used in geometric modeling, like ruled surfaces, canal surfaces or surfaces of revolution, which are generated from a *directrix* or *profile* curve. Furthermore, in geometric modeling it is typical to use rational space curves as profile curves.

In this paper we address the problem of deterministically finding the symmetries of a rational space curve, defined by means of a proper parametrization. Notice that since we deal with a global object, i.e., the set of all points in the image of a rational parametrization, and not just a piece of it, the discrete approach from Alt et al. (1988), Brass and Knauer (2004), Jiang et al. (1996), Li et al. (2008) is not suitable here. Determining if a rational space curve is symmetric or not is useful in order to properly describe the topology of the curve (Alcázar and Díaz Toca, 2010). Furthermore, if the space curve is to be used for generating, for instance, a canal surface or a surface of revolution, certain symmetries of the curve will be inherited by the generated surface. Hence, for modeling purposes it can be interesting to know these symmetries in advance.

Recently, the problem of determining whether a rational plane or space curve is symmetric has been addressed in Alcázar (2014), Alcázar et al. (2014a, 2014b) using a different approach. The common denominator in these papers is the following observation: If a rational curve is *symmetric*, i.e., invariant under a nontrivial isometry f, then this symmetry induces another parametrization of the curve, different from the original parametrization. Assuming that the initial parametrization is proper (definition below), the second parametrization is also proper. Since two proper parametrizations of the same curve are related by a Möbius transformation (Sendra et al., 2008), determining the symmetries is reduced to finding this transformation, therefore translating the problem to the parameter space. This observation leads to algorithms for determining the symmetries of plane curves with polynomial parametrizations (Alcázar, 2014) and of plane and space curves with rational parametrizations (Alcázar et al., 2014b), although in the latter case of general space curves only involutions were considered. The more general problem of determining whether two rational plane curves are similar was considered in Alcázar et al. (2014a).

In this paper we again employ the above observation, but in addition we now also use well-known differential invariants of space curves, namely the curvature and the torsion. The improvement over the method in Alcázar et al. (2014b) is threefold: First of all, we are now able to find *all* the symmetries of the curve instead of just the involutions. Secondly, the new algorithm is considerably faster and can efficiently handle even curves with high degrees and large coefficients in reasonable timings. Finally, the method is simpler to implement and requires fewer assumptions on the parametrization.

Some general facts on symmetries of rational curves are presented in Section 2. Section 3 provides an algorithm for checking whether a curve is symmetric. The determination of the symmetries themselves is addressed in Section 4. Finally, in Section 5 we report on the performance of the algorithm, by presenting a complexity analysis and providing timings for several examples, including a comparison with the curves tested in Alcázar et al. (2014b).

2. Symmetries of rational curves

Throughout the paper, we consider a rational space curve $\mathcal{C} \subset \mathbb{R}^3$, neither a line nor a circle, parametrized by a rational map

$$\boldsymbol{x} : \mathbb{R} \dashrightarrow \mathcal{C} \subset \mathbb{R}^3, \qquad \boldsymbol{x}(t) = \big(x(t), y(t), z(t)\big). \tag{1}$$

The components $x(t), y(t), z(t)$ of $\boldsymbol{x}$ are rational functions of t with rational coefficients, and they are defined for all but a finite number of values of t. Let the *(parametric)* degree m of $\boldsymbol{x}$ be the maximal degree of the numerators and denominators of the components $x(t), y(t), z(t)$. Note that rational curves are irreducible. We assume that the parametrization (1) is *proper*, i.e., birational or, equivalently, injective except for perhaps finitely many values of t. This can be assumed without loss of generality, since any rational curve can quickly be properly reparametrized. For these claims and other results on properness, the interested reader can consult (Sendra et al., 2008) for plane curves and (Alcázar, 2012, §3.1) for space curves.

We recall some facts from Euclidean geometry (Coxeter, 1969). An *isometry* of $\mathbb{R}^3$ is a map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ preserving Euclidean distances. Any isometry f of $\mathbb{R}^3$ is linear affine, taking the form

$$f(\boldsymbol{x}) = \boldsymbol{Q}\,\boldsymbol{x} + \boldsymbol{b}, \qquad \boldsymbol{x} \in \mathbb{R}^3, \tag{2}$$

with $\boldsymbol{b} \in \mathbb{R}^3$ and $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$ an orthogonal matrix. In particular $\det(\boldsymbol{Q}) = \pm 1$. Under composition, the isometries of $\mathbb{R}^3$ form the *Euclidean group*, which is generated by *reflections*, i.e., symmetries with respect to a plane, or *mirror symmetries*. An isometry is called *direct* when it preserves the orientation, and *opposite* when it does not. In the former case $\det(\boldsymbol{Q}) = 1$, while in the latter case $\det(\boldsymbol{Q}) = -1$. The identity map of $\mathbb{R}^3$ is called the *trivial symmetry*.

The classification of the nontrivial isometries of Euclidean space includes reflections (in a plane), rotations (about an axis), and translations, and these combine in commutative pairs to form twists, glide reflections, and rotatory reflections. Composing three reflections in mutually perpendicular planes through a point $\boldsymbol{p}$ yields a *central inversion* (also called *central symmetry*), with center $\boldsymbol{p}$, i.e., a symmetry with respect to the point $\boldsymbol{p}$. The particular case of rotation by an angle π is of special interest, and it is called a *half-turn*. Rotation symmetries are direct, while mirror and central symmetries are opposite.