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Bounding and estimating the Hausdorff distance between real space algebraic curves ☆

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ABSTRACT

In this paper, given two real space algebraic curves, not necessarily bounded, whose Hausdorff distance is finite, we provide bounds of their distance. These bounds are related to the distance between the projections of the space curves onto a plane (say, $z = 0$), and the distance between the z-coordinates of points in the original curves. Therefore, we provide a theoretical result that reduces the estimation and bounding of the Hausdorff distance of algebraic curves from the spatial to the planar case. Using these results we provide an estimation method for bounding the Hausdorff distance between two space curves and we check in applications that the method is accurate and fast.

1. Introduction

The Hausdorff distance has proven to be an appropriate tool for measuring the resemblance between two geometric objects, becoming in consequence a widely used tool in computer aided design, pattern matching and pattern recognition (see for instance Bai et al., 2011; Chen et al., 2010; Kim et al., 2010 and Patrikalakis and Maekawa, 2001). Several variants of the Hausdorff distance have been developed to match specific patterns of objects, in this paper, we study the computation of the Hausdorff distance between two real space algebraic curves $\mathcal{E}_1$ and $\mathcal{E}_2$. We briefly recall the notion of Hausdorff distance; for further details we refer to Aliprantis and Border (2006). In a metric space (X, d), for $\emptyset \neq B \subset X$ and $a \in X$ we define $\mathrm{d}(a, B) = \inf_{b\in B}\{\mathrm{d}(a, b)\}$. Moreover, for $A, B \subset X \setminus \emptyset$ we define

$$\mathrm{H_d}(A, B) = \max\bigl\{\sup_{a\in A}\bigl\{\mathrm{d}(a, B)\bigr\}, \sup_{b\in B}\bigl\{\mathrm{d}(b, A)\bigr\}\bigr\}.$$

By convention $\mathrm{H_d}(\emptyset, \emptyset) = 0$ and, for $\emptyset \neq A \subset X$, $\mathrm{H_d}(A, \emptyset) = \infty$. The function $\mathrm{H_d}$ is called the Hausdorff distance induced by d. In our case, since we will be working in $(\mathbb{C}^3, \mathrm{d})$ or $(\mathbb{R}^3, \mathrm{d})$, d being the usual unitary or Euclidean distance, we simplify the notation writing $\mathrm{H}(A, B)$.

The problem of computing the Hausdorff distance has proven not to be an easy one. We should note in the first place that there is no effective algorithm for the exact computation of the Hausdorff distance between algebraic varieties. Some recent works that approach special cases are Chen et al. (2010), Jüttler (2000) and Patrikalakis and Maekawa (2001). There

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exist theoretical results, as Lojasiewicz inequality for the compact case (see Ji et al., 1992), that relate by means of a constant the Hausdorff distance to the evaluation of the implicit equation(s) of one of the varieties at a parametrization of the other variety. However, this constant is hard to compute. Furthermore, if both varieties are given in implicit form, the computation of the Hausdorff distance is even harder. Also, for the compact case, there are techniques to approximate the distance into a fixed size frame (see Belogay et al., 1997), or by using biarcs (see Kim et al., 2010), or polylines (see Bai et al., 2011) as well as under the phenomenon where the point sets are given imprecisely (see Knauer et al., 2011). Additionally, for plane curves, in Jüttler (2000) it is shown how to bound the Hausdorff distance using foot-point distance. Altogether shows that bounding and estimating the Hausdorff distance is an active research area.

Among all these different variants of the problem we, here, deal with the algebraic global case for space curves. That is, we are given two real space algebraic curves and we want to provide bounds of the distance of the two algebraic sets and not of certain parts (subsets) included within a bounded frame. Afterwards, these bounds can be used to obtain estimations of the Hausdorff distance in a chosen bounded frame. Having such bounds would be useful for measuring the performance of approximate parametrization and approximate implicitization methods, see Emiris et al. (2012), Pérez-Díaz et al. (2010), Rueda and Sendra (2012), Rueda et al. (2013), in other words to decide how much the input and output of such methods resemble each other. In Pérez-Díaz et al. (2010) we provided bounds for the Hausdorff distance between two algebraic plane curves and, in practice, we used estimations of these bounds in Rueda and Sendra (2012). In Rueda et al. (2013), we gave a method to estimate bounds of $\sup_{P\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(P,\mathcal{E}_2)\}$ for space curves, considering the intersection of normal planes through regular points of $\mathcal{E}_1^{\mathbb{R}}$ with $\mathcal{E}_2$, and conversely; the curves, although real, are considered over the field of complex numbers and the super-index $\mathbb{R}$ means the real part of the curve.

In this paper, considering planes orthogonal to a plane of projection ($z=0$), we will be able to relate bounds of $\sup_{P\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(P,\mathcal{E}_2)\}$ with bounds for the distance between the real parts of the projected curves, as well as with bounds for the distance between the z-coordinates of points in $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2$. To derive this relation a Gröbner basis of the ideal of $\mathcal{E}_2$ w.r.t. the pure lexicographic order with $z>x>y$ is a fundamental tool (see e.g. Cox et al., 1997 for further details on Gröbner bases). The relation between bounds of the space curves and bounds of the plane curves is the main contribution of this paper; see Theorem 3.5 and its corollaries. It allows the use of every method developed so far to estimate the Hausdorff distance between plane curves to achieve estimations of the distance for space curves. In this sense, more profit is obtained from existing techniques for estimating the Hausdorff distance in the plane, since they can be use for the same goal with space curves.

In Section 2, we present a situation in which the Hausdorff distance between the real part of two algebraic curves is finite. Bounds for the distances $\mathrm{d}(P,\mathcal{E}_2)$ and $\mathrm{d}(P,\mathcal{E}_2^{\mathbb{R}})$, $P\in\mathcal{E}_1$ are given in Section 3. In such situation, an estimation method for the supremum of the distances between points in a given curve and the other curve, is given in Section 4. Examples of application of such method are provided in Section 5 and conclusions are derived in Section 6.

2. Notation and general assumptions

In this section we fix the **notation** that will be used throughout the paper. We consider a computable subfield $\mathbb{K}$ of the field $\mathbb{R}$ of real numbers, as well as its algebraic closure $\mathbb{F}$; in practice, we may think that $\mathbb{K}$ is the field $\mathbb{Q}$ of rational numbers. We denote by $\mathbb{F}^2$ and $\mathbb{F}^3$ the affine plane and the affine space over $\mathbb{F}$, respectively. Similarly, we denote by $\mathbb{P}^2(\mathbb{F})$ and $\mathbb{P}^3(\mathbb{F})$ the projective plane and the projective space over $\mathbb{F}$, respectively. Furthermore, if $\mathcal{A}\subset\mathbb{F}^3$ (similarly if $\mathcal{A}\subset\mathbb{F}^2$) we denote by $\mathcal{A}^*\subset\mathbb{F}^3$ its Zariski closure, and by $\mathcal{A}^h\subset\mathbb{P}^3(\mathbb{F})$ the projective closure of $\mathcal{A}^*$. We will consider (x,y,z) as affine coordinates and $(x:y:z:w)$ as projective coordinates. Also, we denote by $\mathcal{A}^\infty$ the points at infinity of $\mathcal{A}$, that is, the intersection of $\mathcal{A}^h$ with the projective plane (line in the planar case) of equation $w=0$.

In addition, we will consider two irreducible real space curves $\mathcal{E}_1,\mathcal{E}_2\subset\mathbb{C}^3$ satisfying the following assumptions:

A_1. $\mathcal{E}_1^\infty=\mathcal{E}_2^\infty$,
A_2. $\mathrm{card}(\mathcal{E}_1^\infty)=\mathrm{card}(\mathcal{E}_2^\infty)=\deg(\mathcal{E}_1)=\deg(\mathcal{E}_2)$,
A_3. $\mathcal{E}_1,\mathcal{E}_2$ are not included in a plane of the form $ax+by=c$; note that this is not a loss of generality.

We denote by $\mathcal{E}_i^{\mathbb{R}}$ the real part of $\mathcal{E}_i$, that is, $\mathcal{E}_i^{\mathbb{R}}=\mathcal{E}_i\cap\mathbb{R}^3$. Then, in Rueda et al. (2013), Theorem 6.4, it is shown that if assumptions A_1 and A_2 are satisfied then $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}},\mathcal{E}_2^{\mathbb{R}})<\infty$. Alternatively, one may replace assumptions A_1 and A_2 by the requirement that $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2^{\mathbb{R}}$ are both compact, in which case, $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}},\mathcal{E}_2^{\mathbb{R}})<\infty$. In addition, note that the compactness of $\mathcal{E}_i^{\mathbb{R}}$ can be deduced from the structure of the curve at infinity.

In addition to the assumptions above, we consider that the projection of each $\mathcal{E}_i$ over the plane $z=0$ is birational. Note that for almost all projections (see e.g. Fulton, 1989, p. 155), this holds and hence we can assume it w.l.o.g. We denote this projection map by π_z^i to distinguish between the projection restricted to $\mathcal{E}_1$ and to $\mathcal{E}_2$.

We consider a Gröbner basis $\mathcal{F}=\{F_0,F_1,\ldots,F_{\ell_1}\}\subset\mathbb{K}[x,y,z]$ of the ideal of $\mathcal{E}_1$ and a Gröbner basis $\mathcal{G}=\{G_0,G_1,\ldots,G_{\ell_2}\}\subset\mathbb{K}[x,y,z]$ of the ideal of $\mathcal{E}_2$, both bases w.r.t. the pure lexicographic order with $z>x>y$. Let F_0 be the smallest polynomial of $\mathcal{F}$, then $F_0\in\mathbb{K}[x,y]$ and it is an implicit representation of the projected curve $\pi_z^1(\mathcal{E}_1)^*$; similarly with G_0 and $\pi_z^2(\mathcal{E}_2)^*$. On the other hand, since we are assuming the projection π_z^i to be birational on $\mathcal{E}_i$, both Gröbner bases contain a linear polynomial in z. Say that $F_1=f_1(x,y)z-f_2(x,y)$ and that $G_1=g_1(x,y)z-g_2(x,y)$. Note that this