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Detecting symmetries of rational plane and space curves ☆

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ABSTRACT

This paper addresses the problem of determining the symmetries of a plane or space curve defined by a rational parametrization. We provide effective methods to compute the involution and rotation symmetries for the planar case. As for space curves, our method finds the involutions in all cases, and all the rotation symmetries in the particular case of Pythagorean-hodograph curves. Our algorithms solve these problems without converting to implicit form. Instead, we make use of a relationship between two proper parametrizations of the same curve, which leads to algorithms that involve only univariate polynomials. These algorithms have been implemented and tested in the `Sage` system.

1. Introduction

The problem of detecting the symmetries of a curve has been studied extensively, mainly because of its applications in Pattern Recognition, Computer Graphics and Computer Vision.

In Pattern Recognition, a common problem is how to choose, from a database of curves, the one which suits best a given object, represented by means of an equation (Huang and Cohen, 1996; Lei et al., 1998; Sener and Unel, 2005; Tarel and Cooper, 2000; Tasdizen et al., 2000; Taubin, 1991). Before a comparison can be carried out, one must bring the shape that needs to be identified into a canonical position. Thus it becomes necessary to compute the symmetries of the studied curve. In this context, the computation of symmetries has been addressed using splines (Huang and Cohen, 1996), by means of differential invariants (Boutin, 2000; Calabi et al., 1998; Weiss, 1993), using a complex representation of the implicit equation of the curve (Lebmeir and Richter-Gebert, 2008; Lebmeir, 2009; Tarel and Cooper, 2000), and using moments (Huang and Cohen, 1996; Suk and Flusser, 1993, 2005; Taubin and Cooper, 1992).

In Computer Graphics, the detection of symmetries and similarities is important, both in the 2D and the 3D case, to gain understanding when analyzing pictures, and also in order to perform tasks like compression, shape editing or shape completion. Many techniques involve statistical methods and, in particular, clustering; see for example the papers (Berner et al., 2008; Bokeloh et al., 2009; Mitra et al., 2006; Podolak et al., 2006), where the technique of transformation voting is

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used. Other techniques are robust auto-alignment (Simari et al., 2006), spherical harmonic analysis (Martinet et al., 2006), primitive fitting (Schnabel et al., 2008), and spectral analysis (Lipman et al., 2010), to quote a few.

In Computer Vision, symmetry is important for object detection and recognition. In this context, an analysis has been carried out using the Extended Gauss Image (Sun and Sherrah, 1997) and using feature points (Loy and Eklundh, 2006). In addition, there are algorithms for computing the symmetries of 2D and 3D discrete objects (Alt et al., 1988; Brass and Knauer, 2004; Jiang et al., 1996; Li et al., 2008) and for boundary-representation models (Li et al., 2008, 2010; Tate and Jared, 2003).

In the case of discrete objects (polygons, polyhedra), the symmetries can be determined exactly (Alt et al., 1988; Brass and Knauer, 2004; Jiang et al., 1996; Li et al., 2008). This can be generalized to the case of more complicated shapes whose geometry is described by a discrete object, as done in Brass and Knauer (2004), where an efficient algorithm is provided. Examples of this situation appear with Bézier curves and tensor product surfaces, where the shape follows from the geometry of the control points. However, in almost all of the other above references, the goal is to find *approximate* symmetries of the shape. This is perfectly adequate in many applications, because the input is often a 'fuzzy' shape, with missing or occluded parts in some cases. In fact, even if the input is exact, it is often an approximate, simplified model of a real object. Here we shall consider a different perspective. We assume that our input is exact, and we want to deterministically detect the existence and nature of its symmetries, without converting to implicit form. More precisely, our input will be either a plane or a space curve $\mathcal{C}$ defined by means of a rational parametrization with integer coefficients. Our goal is to (1) determine whether $\mathcal{C}$ has any symmetries, and (2) determine all symmetries in the affirmative case.

Notice that since we are dealing with a *global* object, i.e., the whole curve $\mathcal{C}$, we do not have a control polygon from which the geometry of the curve, and in particular its symmetries, can be derived. This could be the case if we were addressing a piece of $\mathcal{C}$, at least when $\mathcal{C}$ admits a polynomial parametrization. In that situation, $\mathcal{C}$ could be brought into Bézier form, and then an algorithm like Brass and Knauer (2004) could be applied. In fact, in that case the algorithm of Brass and Knauer (2004) would be computationally more effective than ours, since essentially the analysis follows from a discrete object. However, this idea is no longer applicable when the whole curve is considered.

Additionally, an analysis of *approximate* symmetries of rational curves could be attempted by sampling points on the curve, and then applying algorithms like Brass and Knauer (2004), Li et al. (2008, 2010). In that case, the question is how to choose suitable zones for sampling, which amounts to collecting some information on the shape of the curve (Alcázar and Díaz-Toca, 2010). A natural strategy is to look for notable points on the curve, like singularities, inflection points or vertices: Since any symmetry maps notable points of a certain nature to the same kind of points or leaves them invariant, one might sample around these points. One thus obtains clusters of points that must be compared. There would be various possibilities for comparing these clusters, depending on the kind of symmetry one is looking for, all which should be explored. Still, this approach only leads to an approximate estimate on the existence of symmetries, which is a different problem than the one considered in this paper.

Up to our knowledge, the deterministic problem for whole curves has only been solved in the case of implicit plane curves (Lebmeir and Richter-Gebert, 2008; Lebmeir, 2009) and in the case of polynomially parametrized plane curves (Alcázar, 2014). The case of space curves seems absent from the literature. In Lebmeir and Richter-Gebert (2008), Lebmeir (2009), the authors provide an elegant method to detect rotation symmetry of an implicitly defined algebraic curve, and efficiently find the exact rotation angle and rotation center. The method uses a complex representation $F(z,\bar{z})=0$ of the curve. Some cases not treated in Lebmeir and Richter-Gebert (2008) are completed in Lebmeir (2009), where similar ideas are applied to detect mirror symmetry. In contrast, our method applies directly to the parametrization, which is the most common representation in CAGD, avoiding the conversion into implicit form. The approach in Alcázar (2014) is similar to ours, although it should be noted that restricting to polynomial parametrizations yields an advantage for solving the problem fast and efficiently.

The main ingredient in our method is the underlying relation between two parametrizations of a curve that are *proper*, i.e., injective except perhaps for finitely many values of the parameter. Essentially, whenever a symmetry is present, this symmetry induces an alternative parametrization of the curve. Furthermore, if the starting parametrization is proper, this second parametrization is also proper. Since two proper parametrizations of a same curve are related by means of a Möbius transformation (Sendra et al., 2008), we can reduce the problem to finding this transformation. Thus, *involutions*, i.e., symmetries with respect to a point, line or plane, can be detected and determined for plane and space curves. For rotations, we need one more ingredient: a formulation in terms of complex numbers for plane curves, or the Pythagorean-hodograph assumption for space curves. In practice, our methods boil down to computing greatest common divisors and finding real roots of univariate polynomials, which are tasks that can be performed efficiently.

2. Symmetries of plane and space curves

Throughout the paper we shall consider a rational curve $\mathcal{C}\subset\mathbb{R}^n$, where $n=2$ or $n=3$, neither a line nor a circle, defined by means of a proper rational parametrization

$$\mathbf{x}:\mathbb{R}\dashrightarrow\mathcal{C}\subset\mathbb{R}^n,\qquad \mathbf{x}(t)=\big(x_1(t),x_2(t),\dots,x_n(t)\big), \tag{1}$$