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Birational quadrilateral maps ☆

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ABSTRACT

A generic planar quadrilateral defines a 2:1 bilinear map. We show that by assigning an appropriate weight to one vertex of any planar quadrilateral, we can create a map whose inverse is rational linear.

1. Introduction

A planar quadrilateral with vertices $\mathbf{P}_{ij} = (x_{ij}, y_{ij})$ defines a bilinear map

$$\mathbf{P}(s,t) = \mathbf{P}_{00}\bar{s}\bar{t} + \mathbf{P}_{10}s\bar{t} + \mathbf{P}_{01}\bar{s}t + \mathbf{P}_{11}st \tag{1}$$

where $\bar{s} = (1-s)$ and $\bar{t} = (1-t)$. If the quad is a trapezoid, the map is 1:1. Otherwise, the map is generally 2:1 and the inverse involves a square root (Wolberg, 1990). For point **E** in Fig. 1.a,

$$s = \frac{x - 4y + 36 \pm \sqrt{f(x,y)}}{24}, \qquad t = \frac{-x + 4y + 36 \pm \sqrt{f(x,y)}}{32},$$

where $f(x,y) = x^2 - 8xy + 16y^2 - 72x - 96y + 1296$. So **E** has pre-images $(s,t) = (\frac{1}{3}, \frac{3}{4})$ and $(s,t) = (2,2)$. Fig. 1.b shows that **E** lies on two different t-isoparameter lines: $t = \frac{3}{4}$ and $t = 2$.

This paper proves that assigning a weight w_{ij} to one control point, the map defined by any non-degenerate quadrilateral can be forced to be generally 1:1 with a rational linear inverse.

For example, in Fig. 2.a we assign $w_{01} = \frac{5}{3}$ to the quad in Fig. 1. In this case, point **E** lies on a single s- and t-isoparameter line. The inversion equations are $s = \frac{3x-4y}{26-4y}$ and $t = \frac{2y}{20-x}$ and the pre-image of **E** is $(s,t) = (\frac{2}{5}, \frac{2}{3})$. This is an example of a birational map, meaning that both the map and its inverse can be expressed as a polynomial divided by a polynomial.

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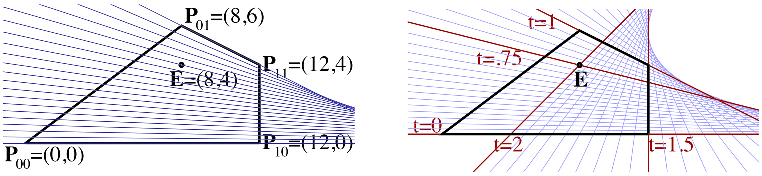
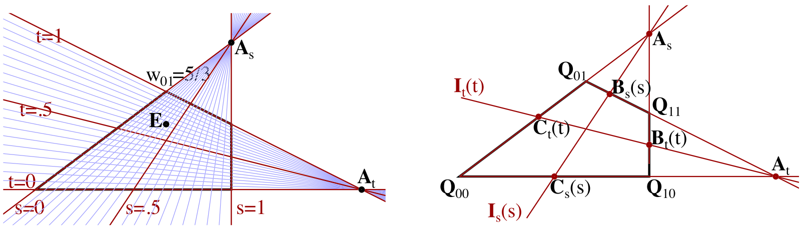


(a) Lines for $0 \le t \le 1$. (b) Lines for $0 \le t \le 2$.

Fig. 1. Quadrilateral with t-isoparameter lines.



(a) Two pencils of lines (b) $Point(\mathbf{Q}_{00})$ is labelled $\mathbf{Q}_{00}$, etc.

Fig. 2. With $w_{01} = \frac{5}{3}$. Both families of isoparameter lines form pencils.

2. Birational quadrilaterals

As illustrated in Fig. 2.a, birational quadrilateral maps are characterized by the fact that each family of isoparameter lines form a pencil, that is, they pivot about axis points $\mathbf{A}_s$ and $\mathbf{A}_t$, respectively. We now show how those pencils can be created by assigning a single control point weight.

Given a triple $\mathbf{Q} = (a, b, c)$ of homogeneous projective coordinates, $Point(\mathbf{Q})$ denotes the point whose Cartesian coordinates are $(a/c, b/c)$ and $Line(\mathbf{Q})$ denotes the line $ax + by + c = 0$. Given triples $\mathbf{Q}_1$ and $\mathbf{Q}_2$, $Line(\mathbf{Q}_1)$ and $Line(\mathbf{Q}_2)$ intersect at $Point(\mathbf{Q}_1 \times \mathbf{Q}_2)$ and $Point(\mathbf{Q}_1)$ and $Point(\mathbf{Q}_2)$ lie on $Line(\mathbf{Q}_1 \times \mathbf{Q}_2)$. The projective coordinates robustly express the intersection of two parallel lines as a point at infinity, i.e., a point for which $c = 0$.

If $\mathbf{Q}_1 \cdot \mathbf{Q}_2 = 0$, $Point(\mathbf{Q}_1)$ lies on $Line(\mathbf{Q}_2)$. If $\mathbf{Q}(t) = (a(t), b(t), c(t))$ is a triple of polynomials, $Point(\mathbf{Q}(t))$ is a rational curve and $Line(\mathbf{Q}(t))$ is called a *moving line* (Sederberg et al., 1994), i.e., a line that moves as a function of t. Denoting $\mathbf{Q}_{ij} = (x_{ij}, y_{ij}, 1)$ and $\tilde{\mathbf{Q}}_{ij} = w_{ij}\mathbf{Q}_{ij}$,

$$\mathbf{Q}(s,t) = Point(\tilde{\mathbf{Q}}_{00}\bar{s}\bar{t} + \tilde{\mathbf{Q}}_{10}s\bar{t} + \tilde{\mathbf{Q}}_{01}\bar{s}t + \tilde{\mathbf{Q}}_{11}st) \tag{2}$$

defines a rational bilinear map. In Fig. 2.b,

$$\begin{aligned}
&\mathbf{A}_s = (\tilde{\mathbf{Q}}_{00} \times \tilde{\mathbf{Q}}_{01}) \times (\tilde{\mathbf{Q}}_{10} \times \tilde{\mathbf{Q}}_{11}), \qquad \mathbf{A}_t = (\tilde{\mathbf{Q}}_{00} \times \tilde{\mathbf{Q}}_{10}) \times (\tilde{\mathbf{Q}}_{01} \times \tilde{\mathbf{Q}}_{11}),\\
&\mathbf{B}_s(s) = \bar{s}\tilde{\mathbf{Q}}_{01} + s\tilde{\mathbf{Q}}_{11}, \qquad \mathbf{C}_s(s) = \bar{s}\tilde{\mathbf{Q}}_{00} + s\tilde{\mathbf{Q}}_{10},\\
&\mathbf{B}_t(t) = \bar{s}\tilde{\mathbf{Q}}_{10} + s\tilde{\mathbf{Q}}_{11}, \qquad \mathbf{C}_t(t) = \bar{s}\tilde{\mathbf{Q}}_{00} + s\tilde{\mathbf{Q}}_{01}.
\end{aligned} \tag{3}$$

For a generic quadrilateral, the families of s- and t-isoparameter lines are $Line(I_s(s))$ and $Line(I_t(t))$, where

$$I_s(s) = \mathbf{B}_s(s) \times \mathbf{C}_s(s), \qquad I_t(t) = \mathbf{B}_t(t) \times \mathbf{C}_t(t).$$

$Line(I_s(s))$ is a pencil with axis $Point(\mathbf{A}_s)$ if $\mathbf{A}_s \cdot I_s(s) \equiv 0$. This implies

$$\begin{aligned}
&w_{01}w_{10}\mathbf{A}_s \cdot \mathbf{Q}_{01} \times \mathbf{Q}_{10} + w_{11}w_{00}\mathbf{A}_s \cdot \mathbf{Q}_{11} \times \mathbf{Q}_{00} = 0 \quad \text{or}\\
&\frac{w_{00}w_{11}}{w_{01}w_{10}} = \frac{\mathbf{A}_s \cdot \mathbf{Q}_{01} \times \mathbf{Q}_{10}}{\mathbf{A}_s \cdot \mathbf{Q}_{00} \times \mathbf{Q}_{11}}.
\end{aligned} \tag{4}$$

Likewise, $Line(I_t(t))$ is a pencil with axis $Point(\mathbf{A}_t)$ if

$$\frac{w_{00}w_{11}}{w_{01}w_{10}} = \frac{\mathbf{A}_t \cdot \mathbf{Q}_{10} \times \mathbf{Q}_{01}}{\mathbf{A}_t \cdot \mathbf{Q}_{00} \times \mathbf{Q}_{11}}. \tag{5}$$

Letting $|ABC|$ denote $A \times B \cdot C$, and applying to (3) the identity

$$(A \times B) \times (C \times D) = \big(A \cdot (B \times D)\big)C - \big(A \cdot (B \times C)\big)D,$$

(4) and (5) become equivalent to