Computer Aided Geometric Design 32 (2015) 50–68

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Strata of rational space curves ☆

David A. Cox [a,*], Anthony A. Iarrobino [b]

[a] Department of Mathematics and Statistics, Amherst College, Amherst, MA 01002, USA
[b] Department of Mathematics, Northeastern University, Boston, MA 02115, USA

ARTICLE INFO

Article history:
Received 13 August 2014
Received in revised form 7 November 2014
Accepted 20 November 2014
Available online 28 November 2014

Keywords:
Rational space curve
Hilbert function
Parametrization
Non-proper maps
$\boldsymbol{\mu}$-strata
Rational normal scroll

ABSTRACT

The $\boldsymbol{\mu}$-invariant $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ of a rational space curve gives important information about the curve. In this paper, we describe the structure of all parameterizations that have the same $\boldsymbol{\mu}$-type, what we call a *$\boldsymbol{\mu}$-stratum*, and as well the closure of strata. Many of our results are based on papers by the second author that appeared in the commutative algebra literature. We also present new results, including an explicit formula for the codimension of the locus of non-proper parametrizations within each $\boldsymbol{\mu}$-stratum and a decomposition of the smallest $\boldsymbol{\mu}$-stratum based on which two-dimensional rational normal scroll the curve lies on.

1. Introduction

A rational curve of degree n in projective 3-space is parametrized by

$$F(s,t) = \big(a_0(s,t), a_1(s,t), a_2(s,t), a_3(s,t)\big) \tag{1.1}$$

where a_0, a_1, a_2, a_3 are relatively prime homogeneous polynomials of degree n. If F is generically one-to-one and a_0, a_1, a_2, a_3 are linearly independent, then the image curve C has degree n and does not lie in a plane, i.e., C is a genuine *space curve*.

For a parametrized planar curve of degree n, Cox et al. (1998) introduced the idea of a *μ-basis*. Since then, μ-bases have proved useful in the study of the singularities of rational plane curves, as evidenced by Chen et al. (2008), Song et al. (2007) in the geometric modeling literature and by Cox et al. (2013) in the commutative algebra literature.

The groundwork for the space curve case appears in Cox et al. (1998, Sec. 5), and the connection with singularities has been studied in several papers, including Jia et al. (2010), Shi and Chen (2010), Shi et al. (2013), Wang et al. (2009). For references to the (fairly extensive) algebraic geometry literature on rational space curves, we direct the reader to the bibliography of Iarrobino (2014).

An idea introduced in Cox et al. (1998) was to study the μ-stratum consisting of *all* parametrizations of plane rational curves with given degree and μ-type. In this paper, we will extend this idea to rational space curves and more generally to define the $\boldsymbol{\mu}$-strata of rational curves in projective d-space, based on the papers by Iarrobino (1977, 2004). A terse version of the results presented in Sections 3 and 4.2, written for commutative algebraists, can be found in Iarrobino (2014). The

☆ This paper has been recommended for acceptance by Falai Chen.

* Corresponding author.
E-mail addresses: dacox@amherst.edu (D.A. Cox), a.iarrobino@neu.edu (A.A. Iarrobino).

http://dx.doi.org/10.1016/j.cagd.2014.11.004
0167-8396/

results concerning properness in Section 4.1 are new to this paper and this topic is not mentioned in Iarrobino (1977, 2004). Results concerning the decomposition of the smallest $\boldsymbol{\mu}$-stratum are geometric consequences of Iarrobino (2004), but were not developed there.

We will review the planar case in Section 2 and then discuss $\boldsymbol{\mu}$-strata in higher dimensions in Section 3. We will give examples to illustrate the unexpected behaviors that can arise. Section 4 will study non-proper parametrizations and explain how parametrizations in the smallest $\boldsymbol{\mu}$-stratum relate to two-dimensional rational normal scrolls in $\mathbb{P}^d$. Proofs will be given in Appendix A.

2. Planar rational curves

For the rest of the paper, we will work over an arbitrary infinite field k, which in practice is usually $\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$. Set $R = \mathsf{k}[s,t]$ and let R_n be the subspace consisting of homogeneous polynomials of degree n.

A rational curve in the projective plane is parametrized by

$$F(s,t) = \big(a_0(s,t), a_1(s,t), a_2(s,t)\big), \tag{2.1}$$

where $a_0, a_1, a_2 \in R_n$. In this section, we will assume that a_0, a_1, a_2 are relatively prime and linearly independent and that F is generically one-to-one.

A *moving line* of degree m is a polynomial $A_0(s,t)x + A_1(s,t)y + A_2(s,t)z$ with $A_0, A_1, A_2 \in R_m$. It *follows* the parametrization if

$$A_0a_0 + A_1a_1 + A_2a_2 = 0 \quad \text{in } R. \tag{2.2}$$

A μ*-basis* for F consists of a pair of moving lines p, q that follow the parametrization and have the property that *any* moving line that follows the parametrization is a linear combination (with polynomial coefficients) of p and q. Assuming $\deg(p) \le \deg(q)$, one sets $\mu = \deg(p)$, so that $\deg(q) = n - \mu$ since it is known that $\deg(p) + \deg(q) = n$. In this situation, we say that F has *type* μ (this is the terminology used in Shi et al., 2013). Thus the μ-type is the minimum degree of a moving line that follows F. Note that

$$1 \le \mu \le \lfloor n/2 \rfloor$$

since a_0, a_1, a_2 are linearly independent and $\mu \le n - \mu$.

To connect this with algebraic geometry and commutative algebra, we introduce the ideal $I = \langle a_0, a_1, a_2 \rangle \subseteq R$. Then, as explained in Cox et al. (1998), the Hilbert Syzygy Theorem gives an exact sequence

$$0 \longrightarrow R(-n-\mu) \oplus R(-2n+\mu) \xrightarrow{A} R(-n)^3 \xrightarrow{B} I \longrightarrow 0, \tag{2.3}$$

where B is given by (a_0, a_1, a_2) and A is the 3×2 matrix whose columns are the coefficients of p and q, and $BA = 0$. The notation $R(-n-\mu), R(-2n+\mu), R(-n)$ reflects the degree shifts needed to make the maps in (2.3) preserve degrees. We note that μ-bases and μ-types appeared in the algebraic geometry literature as early as 1986 (see Ascenzi, 1986, 1988).

When we think of p and q as columns of the matrix A, then the Hilbert–Burch Theorem asserts that the cross product $p \times q$ is the parametrization (a_0, a_1, a_2), up to multiplication by a nonzero constant. This feature makes it easy to create parametrizations with given μ-type: just choose generic p and q of respective degrees μ and $n - \mu$ and take their cross product.

To study all parametrizations with the same μ-type, we begin with the subset $\mathcal{P}_n \subseteq R_n^3$ consisting of all relatively prime linearly independent triples (a_0, a_1, a_2) for which the parametrization is generically one-to-one. Then, for $1 \le \mu \le \lfloor n/2 \rfloor$, we have the μ*-stratum*

$$\mathcal{P}_n^\mu = \big\{(a_0, a_1, a_2) \in \mathcal{P}_n \;\big|\; (a_0, a_1, a_2) \text{ has type } \mu\big\}.$$

In Cox et al. (1998), it was proved that $\mathcal{P}_n^\mu$ is irreducible of dimension

$$\dim\big(\mathcal{P}_n^\mu\big) = \begin{cases} 3(n+1), & \text{if } \mu = \lfloor n/2 \rfloor, \\ 2n + 2\mu + 4, & \text{if } \mu < \lfloor n/2 \rfloor. \end{cases} \tag{2.4}$$

The μ-stratum $\mathcal{P}_n^\mu$ is not closed in $\mathcal{P}_n$. Let $\overline{\mathcal{P}}_n^\mu$ denote its Zariski closure in $\mathcal{P}_n$. In Cox et al. (1998), it was conjectured that

$$\overline{\mathcal{P}}_n^\mu = \mathcal{P}_n^1 \cup \cdots \cup \mathcal{P}_n^\mu. \tag{2.5}$$

This was proved by D'Andrea (2004). This result also is a consequence of the memoir by Iarrobino (1977) or the article by Iarrobino (2004), though these are written from a very different viewpoint. It was eight years after D'Andrea (2004) appeared before a connection was made between these papers.