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On the maximum angle condition for the conforming longest-edge n-section algorithm for large values of n ☆

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ABSTRACT

In this note we introduce the conforming longest-edge n-section algorithm and show that for $n \geq 4$ it produces a family of triangulations which does not satisfy the maximum angle condition.

1. Introduction

The classical longest-edge (LE) bisection algorithm bisects simultaneously all triangles by medians to the longest edge of each triangle in a given triangulation. In this way, an infinite sequence of nested triangulations can be generated. However, this type of refinements may lead, in general, to the so-called hanging nodes and thus refined triangulations may not all be conforming, in general (see Fig. 1).

Many real-life applications where triangulations are used, e.g. the calculations by the finite element method (FEM), require the property of conformity (Ciarlet, 1978). Therefore, in Korotov et al. (2008) a modified version of the classical LE-bisection was introduced, where only elements sharing the longest edge of the whole triangulation are bisected at each step (see Fig. 2). In this way, all produced triangulations are conforming a priori and therefore this algorithm is called the *conforming LE-bisection algorithm*. The same idea can, obviously, be used for simplicial meshes in any dimension.

In Suárez et al. (2012), the classical LE-bisection algorithm was generalized in another direction. It was proposed to divide the longest edges into n equal parts (with $n \geq 2$), one calls this technique the *LE n-section*. The performance of this algorithm was analyzed in Plaza et al. (2010) and Suárez et al. (2012) for different values of n when $n \geq 3$. It should be noted that algorithms using the longest-edge trisection of triangles also present the non-degeneracy property of the triangles, that is, the minimum angle is greater than a certain angle away from zero (Plaza et al., 2010), and so the maximum angle condition is also guaranteed. However, as in the classical bisection-version, the LE n-section algorithm may produce hanging nodes.

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Fig. 1. The classical LE-bisection algorithm produces, in general, nonconforming triangulations.

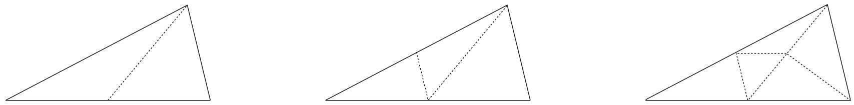

Fig. 2. A modified LE-bisection algorithm that always produces conforming triangulations.

In this work, we blend the ideas of Korotov et al. (2008) and Plaza et al. (2007), Suárez et al. (2012) and define the *conforming LE n-section algorithm* as follows:

a) In the given triangulation we select the longest edge;
b) For two (or one, if the longest edge lies on the boundary) triangles adjacent to this longest edge we apply the LE n-section from Suárez et al. (2012), and thus we generate a new triangulation. If necessary, we go to the step a).

It is clear that we avoid producing hanging nodes by the above defined algorithm in principle. Obviously, just the same idea can be applied to simplicial meshes in any dimension. Notice that the n-section of a single triangle produces n new subedges along divided longest edge, and $n-1$ new subedges in the interior of the triangle, so there finally appear $2n-1$ new subedges in the split triangle. Besides, $3n-2$ new subedges appear when two triangles sharing the longest edge are n-sected.

2. Main results

Let Ω be a bounded polygonal domain with a boundary $\partial\Omega$. In what follows we only deal with conforming triangulations of $\overline{\Omega} := \Omega \cup \partial\Omega$, i.e. an intersection of any two triangles in any triangulation considered is empty, a node, or their adjacent edge. Any triangulation will be denoted by the symbol $\mathcal{T}_h$, where h is the so-called *discretization parameter*, equal to the length of the longest edge in $\mathcal{T}_h$.

Definition 1. The (infinite) sequence of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of $\overline{\Omega}$ is called a *family of triangulations* if for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

In Ženíšek (1969), Zlámal (1968) the following *minimum angle condition* was introduced: there should exist a constant α_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$0 < \alpha_0 \le \alpha_K, \tag{1}$$

where α_K is the minimal angle of K. Under this condition various a priori error estimates for the finite element method (FEM) applied to some elliptic problems are usually derived (Ciarlet, 1978).

Later condition (1) was weakened in Babuška and Aziz (1976), Barnhill and Gregory (1976), Jamet (1976) (see also Křížek, 1991), and the so-called *maximum angle condition* was proposed: There exists a constant γ_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$\gamma_K \le \gamma_0 < \pi, \tag{2}$$

where γ_K is the maximum angle of K.

Remark 1. Condition (1) obviously implies (2), since $\gamma_K \le \pi - 2\alpha_K \le \pi - 2\alpha_0 =: \gamma_0$, but the converse implication does not hold.

In what follows we will prove that for $n \ge 4$ the conforming LE n-section produces triangulations which do not satisfy the maximum angle condition, i.e. there is an infinite sequence of angles in some triangles, among those appearing during the refinement process, which tends to π as the LE n-section proceeds.

Lemma 1. *Let us n-sect the triangle with edges of the length a, b, and c, where $a \le b \le c$. Then there exists a positive constant $\kappa < 1$ such that the lengths of all newly generated sub-edges are not greater than $\kappa \cdot c$.*