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Some properties of LR-splines ☆

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ABSTRACT

Recently a new approach to piecewise polynomial spaces generated by B-spline has been presented by T. Dokken, T. Lyche and H.F. Pettersen, namely Locally Refined splines. In their recent work (Dokken et al., 2013) they define the LR B-spline collection and provide tools to compute the space dimension. Here different properties of the LR-splines are analyzed: in particular the coefficients for polynomial representations and their relation with other properties such as linear independence and the number of B-splines covering each element.

1. Introduction

B-splines are a basis of piecewise polynomials on tensor-product meshes. Their rational counterpart, the NURBS, are the standard tool to represent shapes in Computers Aided Design. This choice was motivated both by the flexibility of B-splines and NURBS for the parametrization of relevant geometric shapes, and by the existence of efficient and stable algorithms to modify, refine and evaluate them. B-splines are available for any polynomial degree and allow different choices of smoothness on the inter-element boundaries. Because of these properties B-splines are a promising choice for the development of Galerkin methods for both ODEs and PDEs. This choice has been proposed in Hughes et al. (2005) under the name of *isogeometric analysis* (the same functions are used for geometry and analysis). Isogeometric analysis has given a new momentum to the research both in the field of geometric design and numerical analysis of PDEs. A more comprehensive description of isogeometric analysis can be found in Cottrell et al. (2009). The main advantages of the isogeometric method are:

- the NURBS geometry is exactly represented within the solver, and is preserved by refinements;
- the refinement of the space is flexible: both p-, h- and k-refinement are possible, the latter being a combination of the first two (Hughes et al., 2005);
- it is possible to use highly smooth functions that enjoy superior approximation properties (Evans et al., 2009; Beirão da Veiga et al., 2011) and are the foundation of many isogeometric analysis schemes (Buffa et al., 2011a, 2011b; Evans and Hughes, 2012a, 2012b).

The weakness of the B-spline basis, both as design and as analysis tool, is the tensor-product structure of the underlying mesh that hinders local refinements, and thus forces the use of big discrete spaces and causes a loss in efficiency. In the last two decades there were many attempts to overcome this limitation: in 1988 Forsay and Bartels introduced the hierarchical-splines (Forsey and Bartels, 1988), in 2003 Sederberg introduced T-splines (Sederberg et al., 2003, 2004) (see

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also the extension in Finnigan, 2008), in 2008 Deng, Chen, Li, Hu, Tong, Yang and Feng introduced PHT-splines (Deng et al., 2008), and in 2012 Dokken, Lyche, and Pettersen introduced LR-splines (Dokken et al., 2013). T-splines are the currently preferred framework for refinement in isogeometric analysis (Bazilevs et al., 2010). T-splines were initially introduced for Computer Aided Design and, at the beginning, their use in analysis has shown some difficulties. Indeed the generators can be linearly dependent (Buffa et al., 2010), they may produce severe fill-in of the mesh (Dörfel et al., 2010) and their properties are difficult to study. These problems were and are being addressed by researchers. For bivariate T-splines, most of these problems were solved by identifying a subset of T-meshes for which some fundamental properties hold. This subset is called *analysis suitable T-splines* (Scott et al., 2012; da Beirão et al., 2012; Li et al., 2012; Beirão da Veiga et al., 2012; Li and Scott, 2012). The development of LR-splines started in parallel to find an alternative refinement strategy. Contrarily to AS T-splines, LR-splines are defined from the description of the discrete space in terms of Bezier domains and continuity conditions. This point of view makes the theory more structured and organized.

The present manuscript extends the work in Dokken et al. (2013) and provides some theoretical results about the LR B-spline collection. The following issues are addressed:

1. the number of LR-splines with overlapping support;
2. the partition of unity property of LR-splines;
3. the adaptation of the blossoming formula for the representation of polynomials and the construction of quasi-interpolants on LR-splines spaces.

In particular here it is proved that if there are not two B-splines in the LR B-spline collection whose supports are nested then the number of B-spline on each element equals the space dimension, moreover the LR B-spline collection is a partition of unity and the coefficients for polynomial representation are the same as in tensor product case. Actually all the properties above are equivalent to the more technical condition that there are no *nested* (see Definition 2.4) B-splines in the LR B-spline collection.

These properties have a clear impact on the use of LR-splines in isogeometric analysis applications. E.g., from 1 it is possible to obtain a bound for the number of nonzero entries in mass and stiffness matrices in applications to PDEs and from 3 it is possible to define quasi-interpolants on the space spanned by the LR B-spline collection.

This text is structured in the following way: Section 2 contains preliminary material about B-splines and box-meshes; this is a review of the content of Dokken et al. (2013). Section 3 contains the study of the overlap, that is the number of nonzero B-spline in each point of the domain. Section 4 deals with the coefficients of polynomial representations by reducing it to the tensor-product case. Section 5 summarizes the previous results and link them together.

Note that all the results are valid for general box-meshes and not only for LR-meshes.

2. Preliminaries and notation

This section contains a selection of the concepts and results from Dokken et al. (2013). There are few modifications and additions that are due the different focus. They are always highlighted and commented.

2.1. B-splines

This subsection shows the used notation for B-splines and introduces the concept of *nested* B-splines.

A B-spline function $B_\Xi : \mathbb{R} \to \mathbb{R}$ of degree p is a right continuous piecewise polynomial of degree p uniquely identified by a knot vector $\Xi = (\xi_1 \leqslant \cdots \leqslant \xi_{p+2})$ such that $\xi_1 < \xi_{p+2}$. Given ζ the multiplicity of ζ in Ξ is $m_\Xi(\zeta) = \#\{j\colon\ \xi_j = \zeta\}$ and it is related with the regularity of B_Ξ in ζ. Indeed B_Ξ is $C^{p-m_\Xi(\zeta)}$ on ζ. The support of B_Ξ is $\operatorname{supp} B_\Xi = [\xi_1, \xi_{p+2}]$. Multivariate B-splines are obtained by multiplying univariate B-splines. In particular $B_{\boldsymbol{\Xi}} : \mathbb{R}^d \to \mathbb{R}$ with $\boldsymbol{\Xi} = (\Xi_1, \ldots, \Xi_d)$ and $\Xi_i = (\xi_{i,1} \leqslant \cdots \leqslant \xi_{i,p_i+2})$ is the B-spline of degree $\mathbf{p} = (p_1, \ldots, p_d)$ defined by

$$B_{\boldsymbol{\Xi}}(\mathbf{x}) = \prod_{i=1}^{d} B_{\Xi_i}(x_i).$$

The basic B-spline refinement method is *knot insertion* and consists in adding a new knot to the knot vector in order to obtain two new knots vectors of the same length. Here the knot insertion is seen as an operation on B-splines (it is described as knot vector refinement in Dokken et al., 2013).

Definition 2.1. The *knot insertion operator* associated to a direction $k \in \{1, \ldots, d\}$ is $R_k : \{\text{B-splines} : \mathbb{R}^d \to \mathbb{R}\} \times \mathbb{R} \times \{1, 2\} \to \{\text{B-splines} : \mathbb{R}^d \to \mathbb{R}\}$ defined by

$$R_k(B_{\boldsymbol{\Xi}}, \bar{\xi}, \sigma) = \begin{cases} B_{\boldsymbol{\Xi}}, & \bar{\xi} \notin \,]\xi_{k,1}, \xi_{k,p_k+2}[, \\ B_{\boldsymbol{\Theta}_\sigma}, & \text{otherwise} \end{cases}$$