Computer Aided Geometric Design 29 (2012) 99–108

Contents lists available at SciVerse ScienceDirect

Computer Aided Geometric Design

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Intersection curves of hypersurfaces in $\mathbb{R}^4$ ☆

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ARTICLE INFO

Article history:
Received 21 May 2011
Accepted 26 October 2011
Available online 9 November 2011

Keywords:
Geometric properties
Frenet frame
Frenet apparatus
Hypersurface
Surface–surface–surface intersection
Transversal intersection
Curvatures

ABSTRACT

We present algorithms for computing the differential geometry properties of Frenet apparatus $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}_1, \boldsymbol{b}_2, \kappa_1, \kappa_2, \kappa_3)$ of intersection curves of implicit–parametric–parametric and implicit–implicit–parametric hypersurfaces in $\mathbb{R}^4$, for transversal intersection. Some examples are given and plotted.

1. Introduction

The intersection problem is a fundamental process needed in modeling complex shapes in CAD/CAM system. It is useful in the representation of the design of complex objects, in computer animation and in NC machining for trimming off the region bounded by the self-intersection curves of offset surfaces. It is also essential to Boolean operations necessary in the creation of boundary representation in solid modeling (Ye and Maekawa, 1999). The numerical marching method is the most widely used method for computing intersection curves in $\mathbb{R}^3$. The Marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local differential geometry (Bajaj et al., 1988; Patrikalakis, 1993). While differential geometry of a parametric curve in $\mathbb{R}^3$ can be found in textbooks such as in Struik (1950), Willmore (1959), Stoker (1969), do Carmo (1976), differential geometry of a parametric curve in $\mathbb{R}^n$ can be found in textbook such as in the contemporary literature on Geometric Modeling (Farin, 2002; Hoschek and Lasser, 1993), there is little literature on differential geometry of intersection curves in $\mathbb{R}^3$ and, rarely, in $\mathbb{R}^4$ (Aléssio, 2009). Willmore (1959) and Aléssio (2006) described how to obtain the unit tangent, the unit principal normal, the unit binormal, the curvature and the torsion of the transversal intersection curve of two implicit surfaces (Düldül, 2010). Kruppa (1957) explained that the tangential direction of the intersection curve at a tangential intersection point corresponds to the direction from the intersection point towards the intersection of the Dupin indicatrices of the two surfaces. Hartmann (1996) provided formulas for computing the curvature of the transversal intersection curves for all types of intersection problems in Euclidean 2-space. Kriezis et al. (1992) determined the marching direction for tangential intersection curves based on the fact that the determinant of the Hessian matrix of the oriented distance function is zero. Luo et al. (1995) presented a method to trace such tangential intersection curves for parametric–parametric surfaces employing the marching method. The marching direction is obtained by solving an undetermined system based on the equilibrium of the differentiation of the two normal vectors and the projection of the Taylor expansion of the two surfaces onto the normal vector at the intersection point.

☆ This paper has been recommended for acceptance by Ron Goldman.

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2011.10.004

Ye and Maekawa (1999) presented algorithms for computing all the differential geometry properties of both transversal and tangentially intersection curves of two parametric surfaces. They described how to obtain these properties for two implicit surfaces or parametric–implicit surfaces. They also gave algorithms to evaluate the higher-order derivative of the intersection curves. Goldman (2005) provides formulas for computing the curvature κ and torsion τ of intersection curve of two implicit surfaces in $\mathbb{R}^3$. Aléssio (2009) and Düldül (2010) studied the differential geometry properties of intersection curves of three implicit surfaces and three parametric surfaces in $\mathbb{R}^4$, respectively. Our previous work (Soliman et al. (August 2011)) presented algorithms for computing differential geometry properties of both transversal and tangentially intersection curves of implicit and parametric surfaces in $\mathbb{R}^3$.

This study is based on the recent papers of Ye and Maekawa (1999), Aléssio (2009), Düldül (2010) and our previous work (August 2011). We present algorithms for computing all the differential geometry properties of intersection curves $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}_1, \boldsymbol{b}_2, \kappa_1, \kappa_2, \kappa_3)$ of implicit–parametric–parametric and implicit–implicit–parametric hypersurfaces in $\mathbb{R}^4$ for transversal intersection. Finally some examples are given and plotted.

2. Geometric preliminaries

Let us first introduce some notations and definitions. Bold letters such as $\boldsymbol{a}$, $\boldsymbol{R}$ will be used for vectors and vector functions. We assume that they are smooth enough so that all the (partial) derivatives given in the paper are meaningful. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be the standard basis of the four-dimensional Euclidean space $\mathbb{R}^4$. The cross product of three vectors $\boldsymbol{a} = \sum_{i=1}^4 a_i \boldsymbol{e}_i$, $\boldsymbol{b} = \sum_{i=1}^4 b_i \boldsymbol{e}_i$, and $\boldsymbol{c} = \sum_{i=1}^4 c_i \boldsymbol{e}_i$ is defined by

$$\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}.$$

The dot (scalar) product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^4 a_i b_i.$$

The length of a vector $\boldsymbol{a}$ is defined by

$$\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}.$$

2.1. Regular curve in $\mathbb{R}^4$

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^4$ be a regular unit speed curve in $\mathbb{R}^4$,

$$\boldsymbol{\alpha}(s) = \big(x_1(s), x_2(s), x_3(s), x_4(s)\big). \tag{2.1}$$

The notation for the differentiation of a curve $\boldsymbol{\alpha}$ in relation to the arc length s is $\boldsymbol{\alpha}'(s) = \frac{d\boldsymbol{\alpha}}{ds}$, $\boldsymbol{\alpha}''(s) = \frac{d^2\boldsymbol{\alpha}}{ds^2}$, $\boldsymbol{\alpha}'''(s) = \frac{d^3\boldsymbol{\alpha}}{ds^3}$, $\boldsymbol{\alpha}^{(4)}(s) = \frac{d^4\boldsymbol{\alpha}}{ds^4}$ and $' = \frac{d}{ds}$.

From elementary differential geometry, we have

$$\begin{aligned} &\boldsymbol{\alpha}' = \boldsymbol{t}, \qquad \boldsymbol{\alpha}'' = \kappa_1 \boldsymbol{n}, \qquad \boldsymbol{\alpha}''' = -(\kappa_1)^2 \boldsymbol{t} + \kappa_1' \boldsymbol{n} + \kappa_1 \kappa_2 \boldsymbol{b}_1, \\ &\boldsymbol{\alpha}^{(4)} = -3\kappa_1 \kappa_1' \boldsymbol{t} + \big(-(\kappa_1)^3 + \kappa_1'' - \kappa_1 (\kappa_2)^2\big)\boldsymbol{n} + \big(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\big)\boldsymbol{b}_1 + \kappa_1 \kappa_2 \kappa_3 \boldsymbol{b}_2, \end{aligned} \tag{2.2}$$

where $\boldsymbol{t}$ is the unit tangent vector field of $\boldsymbol{\alpha}$, $\boldsymbol{n}$ is the unit principal normal vector field of $\boldsymbol{\alpha}$, $\boldsymbol{b}_1$ is the first unit binormal vector field of $\boldsymbol{\alpha}$, $\boldsymbol{b}_2$ is the second unit binormal vector field of $\boldsymbol{\alpha}$ and κ_i $(i = 1, 2, 3)$ are the ith curvature functions of the curve $\boldsymbol{\alpha}$.

The Frenet frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}_1, \boldsymbol{b}_2\}$ along $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{t} = \boldsymbol{\alpha}', \qquad \boldsymbol{n} = \frac{\boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}''\|}, \qquad \boldsymbol{b}_2 = \frac{\boldsymbol{\alpha}' \otimes \boldsymbol{\alpha}'' \otimes \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \otimes \boldsymbol{\alpha}'' \otimes \boldsymbol{\alpha}'''\|}, \qquad \boldsymbol{b}_1 = \boldsymbol{b}_2 \otimes \boldsymbol{t} \otimes \boldsymbol{n}. \tag{2.3}$$

The Frenet–Serret formulas along $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{t}'(s) = \kappa_1 \boldsymbol{n}, \qquad \boldsymbol{n}'(s) = -\kappa_1 \boldsymbol{t} + \kappa_2 \boldsymbol{b}_1, \qquad \boldsymbol{b}_1'(s) = -\kappa_2 \boldsymbol{n} + \kappa_3 \boldsymbol{b}_2, \qquad \boldsymbol{b}_2'(s) = -\kappa_3 \boldsymbol{b}_1. \tag{2.4}$$

Using (2.3) and (2.4) it is easy to see that the curvature functions of $\boldsymbol{\alpha}$ are given by

$$\kappa_1 = \sqrt{\langle \boldsymbol{\alpha}'', \boldsymbol{\alpha}'' \rangle}, \qquad \kappa_2 = \frac{\langle \boldsymbol{\alpha}''', \boldsymbol{b}_1 \rangle}{\kappa_1}, \qquad \kappa_3 = \frac{\langle \boldsymbol{\alpha}^{(4)}, \boldsymbol{b}_2 \rangle}{\kappa_1 \kappa_2}. \tag{2.5}$$