Computer Aided Geometric Design 29 (2012) 150–161

Contents lists available at SciVerse ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

L-system specification of knot-insertion rules for non-uniform B-spline subdivision ☆

V. Nivoliers [a,*], C. Gérot [b,**], V. Ostromoukhov [c], N.F. Stewart [d]

[a] *ALICE, Inria, France*
[b] *GIPSA-lab, Grenoble, France*
[c] *LIRIS, CNRS, France*
[d] *LIGUM, Université de Montréal, Canada*

ARTICLE INFO

Article history:
Received 17 March 2011
Received in revised form 24 November 2011
Accepted 29 November 2011
Available online 7 December 2011

Keywords:
L-system
Subdivision
Non-uniform

ABSTRACT

Subdivision schemes are based on a hierarchy of knot grids in parameter space. A univariate grid hierarchy is regular if all knots are equidistant on each level, and irregular otherwise. We use L-systems to design a wide class of systematically described irregular grid hierarchies. Furthermore, we give sufficient conditions on the L-system which guarantee that the subdivision scheme, based on the non-uniform B-spline of degree d defined on the initial knot grid, is uniformly convergent. If n is the number of symbols in the alphabet of the L-system, this subdivision scheme is defined with a finite set of masks (at most n^{d+1}) which does not depend on the subdivision step. We provide an implementation of such schemes which is available as a worksheet for Sage software.

1. Introduction

The representation of smooth objects (functions, curves or surfaces) is a fundamental problem in computer graphics and geometric modelling. One efficient representation consists of parametrising the object with a spline expressed in terms of a B-spline basis. Each polynomial piece of the spline is defined on an interval (or rectangular domain) whose vertices are called knots; the B-spline basis allows us to associate with the knots in parameter space a set of control points in object space, whose positions influence only locally the shape of the object, thus allowing intuitive control.

Subdivision schemes provide an efficient way to draw such smooth objects from given nets of control points and knots, and a given degree (or bi-degree) for the B-spline basis. These schemes consist of inserting new knots and successively computing new control nets, each defining the same smooth object. If the successively inserted knots are dense enough then the control net converges to the smooth object. In practice, a few steps are enough to reach the resolution of a computer screen.

When the knots define a regular grid, often assumed without loss of generality to be $\mathbb{Z}^N$, where N is the dimension of the object, the B-spline basis is uniform. If the new knots are inserted at midpoints, the new B-spline basis remains uniform and the uniform subdivision rules which define the new control net are the same convex combinations, whatever the interval length: they depend only on N and on the degree of the B-spline basis. Such uniform subdivision schemes are the most commonly used.

☆ This paper has been recommended for acceptance by Hartmut Prautzsch.

* Corresponding author.
** Principal corresponding author.
E-mail address: Cedric.Gerot@gipsa-lab.grenoble-inp.fr (C. Gérot).

0167-8396/$ – see front matter
doi:10.1016/j.cagd.2011.11.004

In some cases, however, non-uniform subdivision schemes are needed, either to represent objects which cannot be parametrised with a uniform spline or to deal with knots that do not lie on $2^{-k}\mathbb{Z}^N$ (for example in the context of multi-resolution analysis with irregular samples). We are interested in the non-uniform case: we propose subdivision schemes for B-spline parametrised curves with irregular but controlled knot intervals. In this paper we restrict our attention to univariate schemes ($N = 1$).

Algorithms for inserting one (Boehm, 1980) or several knots (Cohen et al., 1980; Barry and Zhu, 1992) into intervals separating non-equally spaced knots have been known for decades. They lead to subdivision schemes with knot sequences that are possibly completely irregular, and with as many subdivision processes as there are knot intervals in all of the successive subdivision steps.

The method of Non-uniform Recursive Subdivision Surfaces proposed by Sederberg et al. (1998) starts with irregular knot intervals and inserts, at each step, one knot at the midpoint of each interval. This is, therefore, bisection with rules whose coefficients are written as functions of successive interval length, which consequently have to be computed at each subdivision step, for each vertex.

Recently, algorithms which adapt the efficient refine-and-smooth factorisation, proposed by Lane and Riesenfeld (1980) for uniform B-splines, to the non-uniform B-splines, have been proposed (Schaefer and Goldman, 2009; Cashman et al., 2009b). These algorithms define the subdivision as a single process but they require that exactly one knot be inserted in every interval.

The algorithm proposed by Schaefer and Goldman (2009) can be generalised to the insertion of more than one knot in every interval as long as the number of inserted knots is the same whatever the interval within a subdivision step. Cashman et al. (2009a) permit the omission of subdivision of some intervals, but, in order to preserve the locality of the smoothing steps, they do not support the insertion of more than one knot per interval. Whichever algorithm is considered, the positions of inserted knots have to be explicitly given at each subdivision step, necessitating the computation of new coefficients for the rules used at each subdivision step.

Another approach, suggested by Goldman and Warren (1993), considers an affine relationship between the knots of the initial grid and inserts one knot at a constant barycentric position between each pair of adjacent old knots to preserve the affine relationship. The same insertion rule is used at each step, and everywhere in the domain.

We propose a framework to describe a wide class of non-uniform subdivision schemes with knot sequences more irregular than uniform or affine bisection (or trisection, etc.), but with a small set of subdivision processes which does not depend on the subdivision step, and using fixed coefficients that can be computed in advance.

The proposed framework uses context-free L-systems (Herman et al., 1974) to describe the sequences of irregular knot intervals. The specification of uniform schemes for curves (Prusinkiewicz et al., 2002) and surfaces (Velho, 2003), using L-systems, with the goal of easing implementation, has been done previously, by translating known subdivision procedures into the rules of a context-sensitive grammar.

In Section 2 we show that an L-system is ideal for the design of interval subdivision descriptors which are based on lengths of knot intervals rather than on the position of the knots themselves, and which can be used to define non-uniform B-spline subdivision schemes. We also give details on the computation of the subdivision masks. Not every L-system, however, can be used as such an interval subdivision descriptor. For this reason we introduce in Section 3 the concept of a valid L-system, which yields in particular a convergent non-uniform B-spline subdivision scheme, and we introduce sufficient conditions on the rules for an L-system to be valid.

2. Non-uniform subdivision scheme from L-systems

When knots are inserted, the new control points can be expressed as convex combinations of old control points. These combinations depend only on the degree of the B-spline and the two successive sequences of knots. The difference between B-spline subdivision schemes and knot-insertion algorithms is that subdivision inserts many knots at the same time. The cost of these insertions is reduced due to the fact that the necessary data is shared in the computation of the new control points. In uniform subdivision, this sharing is observable in the subdivision mask which collects, for a given control point, the coefficients of its contributions to the subdivided control points. For a subclass of non-uniform subdivision where at most one knot is inserted in every interval, Cashman et al. (2009a) propose a refine-and-smooth implementation which is another way of sharing data within the subdivision process.

Our aim is to give a different subclass of non-uniform B-spline subdivision which allows us to define a finite set of masks that remains the same whatever the subdivision step. Our framework remains non-uniform in the sense that the subdivision rules are not the same everywhere within one subdivision step and may insert different numbers of knots in each interval.

2.1. The L-system as an interval subdivision descriptor

Using an L-system, we describe the subdivision in terms of splitting knot intervals, and we base our formalism on the lengths of the intervals between the knots rather than the position of the knots themselves.

2.1.1. L-systems

A Lindenmayer system or L-system is a particular kind of *rewriting system* where the rules are applied greedily to rewrite as many symbols as possible at each step, whereas usually only one symbol is rewritten at a time. These systems have