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Centroidal Voronoi tessellation in universal covering space of manifold surfaces ☆

Guodong Rong [a,*], Miao Jin [b], Liang Shuai [a], Xiaohu Guo [a]

[a] *Department of Computer Science, University of Texas at Dallas, United States*
[b] *Center for Advanced Computer Studies, University of Louisiana at Lafayette, United States*

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ABSTRACT

The centroidal Voronoi tessellation (CVT) has found versatile applications in geometric modeling, computer graphics, and visualization, etc. In this paper, we first extend the concept of CVT from Euclidean space to spherical space and hyperbolic space, and then combine all of them into a unified framework – the CVT in universal covering space. The novel spherical and hyperbolic CVT energy functions are defined, and the relationship between minimizing the energy and the CVT is proved. We also show by our experimental results that both spherical and hyperbolic CVTs have the similar property as their Euclidean counterpart where the sites are uniformly distributed with respect to given density values. As an example of the application, we utilize the CVT in universal covering space to compute uniform partitions and high-quality remeshing results for genus-0, genus-1, and high-genus (genus > 1) surfaces.

1. Introduction

The Voronoi diagram is a well studied concept in computational geometry, and has a wide usage in different areas in geometric modeling, computer graphics, visualization, etc. (Okabe et al., 1999). The *centroidal Voronoi tessellation* (CVT) is a special case of the Voronoi diagram, where every site coincides with the centroid of its Voronoi cell (Du et al., 1999). The sites in a CVT are uniformly distributed. This property is conjectured by Gersho in 1979 (Gersho, 1979), and has been proved in 2D convex polygons with up to six edges (Fejes Tóth, 2001).

In geometric modeling, many applications require a uniform sampling on a surface, or a partition of a surface where every region covers similar area. These tasks can be achieved simultaneously by computing a CVT on the surface where all sites are constrained on the surface. Such a CVT is usually known as the *constrained CVT* (Du et al., 2003). It is natural to use the geodesic distance to compute the constrained CVT (Peyré and Cohen, 2004), but it is difficult to compute the geodesic distance accurately. Another alternative is to use the 3D Euclidean distance as an approximation (Liu et al., 2009; Rong et al., 2011; Yan et al., 2009), but this may lead to disconnected Voronoi cells if two regions are very close in 3D Euclidean space but are far away along the surface. A better approach is to compute the CVT in a 2D parametrization domain of the surface (Alliez et al., 2005). By assigning appropriate density values, the computed CVT is very close to the constrained CVT computed using the geodesic distance. This method overcomes the shortages of both prior methods, and is more efficient since the computation is performed in a 2D Euclidean domain.

While to parameterize a closed surface to a 2D Euclidean domain, the original surface has to be cut into a genus-0 surface. This makes the sites unable to cross the boundaries in the parametrization domain, and leads to visible artifacts

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* Corresponding author.
E-mail address: guodongrong@utdallas.edu (G. Rong).

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along the cutting edges. In Alliez et al. (2005), a great deal of special care and delicate strategies, such as minimizing the total cutting edge length and matching the cut graph with the feature skeleton, are required. If the cut graph does not coincide much with a set of feature edges, the remeshing results become unacceptable as indicated in Alliez et al. (2005).

This cutting problem can be solved by computing the CVT directly on the *universal covering space* (Klingenberg, 1982) of the surface. For closed genus-1 surfaces, their universal covering space can be embedded in 2D Euclidean space $\mathbb{R}^2$, so the computation of the CVT in the universal covering space is similar as in Alliez et al. (2005) except that sites can move freely across the cutting boundaries. The universal covering space of closed genus-0 surfaces can be embedded in 2D spherical space $\mathbb{S}^2$. As proved later in Section 4, the spherical CVT is identical to the constrained CVT on the sphere. So we can compute the constrained CVT on the sphere to get the uniform samplings on these surfaces. For closed high-genus (genus > 1) surfaces, their universal covering space can be embedded in 2D hyperbolic space $\mathbb{H}^2$. So computing the CVT in hyperbolic space is required and can lead to new geometric modeling techniques for high-genus surfaces.

To the best of our knowledge, the CVT in hyperbolic space has not been studied before. Furthermore, no previous work has systematically studied the CVT in universal covering space. We study the CVT in hyperbolic space in this paper, and combine it with Euclidean CVT and spherical CVT in a unified framework of the CVT in universal covering space.

One difficulty for defining the CVT is how to well define the centroid of a given region in different spaces. In this paper, we extend the *model centroid* (Galperin, 1993) to define the centroid of a Voronoi cell in 2D spherical, Euclidean, and hyperbolic spaces in a unified way. We also prove that the model centroid is in fact the central projection of the centroid in 3D Euclidean space onto the model.

Previous studies on spherical CVT all treat it as the constrained CVT on the sphere. In this paper, we directly define the CVT energy in spherical space and study its relationship with the spherical CVT. We also define the CVT energy in hyperbolic space and prove the relationship between minimizing this energy and the hyperbolic CVT. Following these conclusions, we can prove the convergence of Lloyd's algorithm for both spherical and hyperbolic CVTs. So we can use Lloyd's algorithm to compute them. Based on our extensive experiments, we conjecture the sites in the spherical and hyperbolic CVTs are also uniformly distributed with respect to the corresponding metrics.

We also show how to use the CVT in universal covering space to generate uniform partitions and high quality remeshing results for genus-0, genus-1, and high-genus (genus > 1) surfaces. Compared with previous methods using parametrization in 2D Euclidean space such as (Alliez et al., 2005), the main advantage of using the CVT in universal covering space is that the sites can move freely anywhere on the surface.

The main contributions of this paper include:

- We formally define the CVT energy in spherical space, and prove the relationship between minimizing this energy and the spherical CVT. We also demonstrate the uniformity of the sites in the spherical CVT.
- We extend the concept of CVT into hyperbolic space. We define the CVT energy in hyperbolic space, and prove the relationship between minimizing this energy and the hyperbolic CVT. We also demonstrate the uniformity of the sites in the hyperbolic CVT.
- We prove the convergence of Lloyd's algorithm for spherical and hyperbolic CVTs, and explain the implementation details of using Lloyd's algorithm to compute them.
- We combine spherical, Euclidean, and hyperbolic CVTs into a unified framework – the CVT in universal covering space, and apply it on computing uniform partitions and high quality remeshing results for genus-0, genus-1, and high-genus (genus > 1) surfaces.

The rest of the paper is organized as follows: Section 2 briefly reviews some related previous work. The formal definitions of the CVT in different spaces are given in Section 3, and the corresponding CVT energy functions are defined in Section 4. The relationship between minimizing this energy and the CVT is also proved in Section 4. Section 5 gives details on how to compute the CVT in difference spaces. Section 6 defines the CVT in universal covering space and applies it on geometric modeling applications. Finally, Section 7 concludes the paper with some possible future work.

2. Related work

We briefly review some previous work on how to compute the CVT in Euclidean space. We also list applications of the CVT in geometric modeling.

2.1. Centroidal Voronoi tessellation

The concept of the centroidal Voronoi tessellation was first introduced by Du et al. (1999), but the similar concepts have been studied in different areas long before that, e.g. optimal quantization in signal processing and k-means in pattern recognition.

One of the earliest algorithms to compute the CVT is proposed by MacQueen (1967) which is a probabilistic algorithm. Although the almost sure convergence of this algorithm is proved, its convergence is very slow. Lloyd proposed a deterministic method in 1960s which is officially published later in 1982 (Lloyd, 1982). The convergence of Lloyd's algorithm is later proved by Du et al. (2006). Due to its simplicity and robustness, Lloyd's algorithm is currently the most widely used