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G^1 interpolation by rational cubic PH curves in $\mathbb{R}^3$ ☆

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ABSTRACT

In this paper the G^1 interpolation of two data points and two tangent directions with spatial cubic rational PH curves is considered. It is shown that interpolants exist for any *true* spatial data configuration. The equations that determine the interpolants are derived by combining a closed form representation of a ten parametric family of rational PH cubics given in Kozak et al. (2014), and the Gram matrix approach. The existence of a solution is proven by using a homotopy analysis, and numerical method to compute solutions is proposed. In contrast to polynomial PH cubics for which the range of G^1 data admitting the existence of interpolants is limited, a switch to rationals provides an interpolation scheme with no restrictions.

1. Introduction

Pythagorean-hodograph (or shortly PH) curves form a special subclass of parametric curves. They are characterized by the property that their unit vector field of tangents is rational. Polynomial PH curves thus have a (piecewise) polynomial arc length, planar PH curves possess rational offset curves and spatial PH curves are equipped with rational orthonormal frames. This makes PH curves very useful for many practical applications in CAGD, CAD/CAM systems, CNC machining, robotics, animations, etc. Polynomial PH curves were first introduced in Farouki and Sakkalis (1990) and since then many approximation and interpolation schemes that involve these curves have been developed (see Farouki, 2008 and the references therein). On the other hand, not much work has been done on rational PH curves, since the extension from the polynomial ones is not straightforward. Planar rational PH curves were derived in Pottmann (1995a), Pottmann (1994), and independently in Fiorot and Gensane (1994). Some interpolation schemes involving these curves can be found in Pottmann (1995b).

The first step to spatial rational PH curves is carried out as a short note in Pottmann (1994) based on the dual representation of rational curves. More extensive study can be found in Farouki and Šír (2011), where the construction of spatial rational PH curves is presented and illustrated from a geometric point of view. The spatial rational PH curve is determined by prescribing a rational unit vector field of binormals obtained from a rational unit field of tangents, and a rational function that prescribes the signed distance of the osculating plane from the origin. Rational unit tangents can be constructed using stereographic projection or Euler–Rodrigues frame (ERF) associated with a quaternion polynomial. In Kozak et al. (2014) it is shown, based on the dual approach and ERF, that such a construction leads in general to curves of relatively high degrees. More precisely, the connection between degrees of quaternion polynomials, dual representation and rational PH

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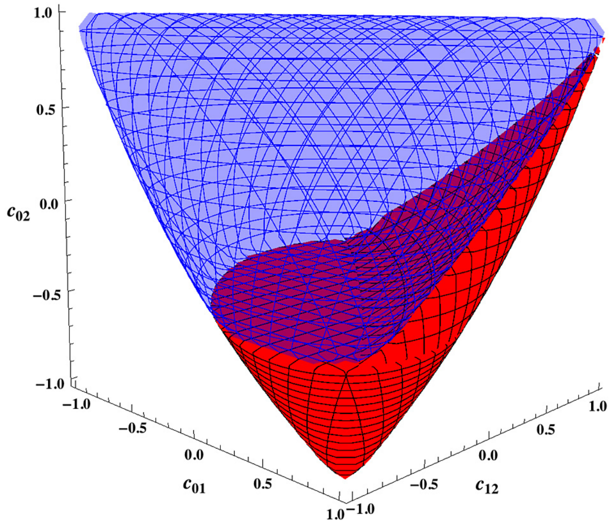

Fig. 1. The red area denotes cosine parameters for which both polynomial and *true* rational solution exist. The blue part has the *true* rational solution only. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

curves is revealed, and the question how to obtain low degree curves is considered in detail. Based on quadratic quaternion polynomials rational PH curves with the dual representation of degree $m = 3, 4, 5, 6$, having $2m + 4$ degrees of freedom are constructed. In particular, cubic rational PH curves with nonconstant denominator depending on ten free parameters are presented in a closed form. Since a family of cubic polynomial PH curves is also ten parametric, rationals do not provide additional degrees of freedom. Moreover both rational and polynomial spatial PH curves have rational canal surfaces (as an equivalent to offset curves in the planar case) and it is also true, that rational PH curves (in contrast to the polynomial ones) do not have rational arc length in general, which might be needed in some motion control applications. However, as shown in this paper, cubic rational PH curves are superior in other aspects to the polynomial ones when used for G^1 interpolation.

The G^1 interpolation of two data points and two tangent directions at these two points by spatial cubic PH curves is important for practical applications and has therefore been considered for polynomial curves in quite a few papers (*e.g.* Wagner and Ravani, 1997; Jüttler and Mäurer, 1999; Pelosi et al., 2005; Kwon, 2010; Jaklič et al., 2012). Since the problem is nonlinear, the analysis of the existence and the number of interpolants according to different G^1 data is quite a difficult task. The conditions that provide the existence of cubic polynomial PH interpolants for many but not all possible data configurations can be found in Wagner and Ravani (1997), Jüttler and Mäurer (1999), Pelosi et al. (2005). A complete characterization of the existence conditions for all possible G^1 data has later been established in Kwon (2010), and further in Jaklič et al. (2012), where nice geometrically intuitive necessary and sufficient conditions on the data have been derived.

Unfortunately, the range of data configurations for which G^1 cubic polynomial PH interpolants do not exist is quite large (see Fig. 1), which is a disadvantage if we want to make a scheme practically useful. In this paper we show that G^1 cubic rational PH interpolants exist for all possible *true* spatial G^1 data. They are determined by nonlinear equations derived by combining a closed form curve representation given in Kozak et al. (2014), and the Gram matrix approach (used also in Jaklič et al., 2012). The existence of the solution is proven by using a homotopy analysis. It is shown that the number of rational interpolants with nonconstant denominator is always odd, and that rational interpolants are separated from the polynomial ones for all possible spatial data. But, as discussed further, if the data are planar, cubic rational PH curves provide the interpolant only in very particular setup. Numerical procedures to compute the solution are proposed too, together with some examples that confirm the theoretical results.

The paper is organized as follows. In the next section the interpolation problem and the main result are presented. Section 3 recalls the quaternion approach from Kozak et al. (2014) and derives some basic properties of interpolants. In Section 4 the Gram matrix approach is presented and in Section 5 the basic PH equations are derived. Homotopy analysis is used in the next section to confirm the existence of a solution for *true* spatial data. Technically demanding, but tedious proofs needed in the section 6 are left to Kozak (2014). In Section 7 the algebraic approach and the planar case are briefly considered. Section 8 discusses the numerical solution of the PH equations. An algorithm is proposed in which one has to find real roots of a couple of single variable degree 6 polynomials only. Numerical examples of the last section conclude the main part of the paper.

2. Interpolation problem

The G^1 interpolation problem is as follows. Suppose that the directions

$$\boldsymbol{d}_0, \boldsymbol{d}_1, \quad \|\boldsymbol{d}_i\| = 1,\ \boldsymbol{d}_i \in \mathbb{R}^3,$$