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Constrained multi-degree reduction with respect to Jacobi norms ☆

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ABSTRACT

We show that a weighted least squares approximation of Bézier coefficients with factored Hahn weights provides the best constrained polynomial degree reduction with respect to the Jacobi L_2-norm. This result affords generalizations to many previous findings in the field of polynomial degree reduction. A solution method to the constrained multi-degree reduction with respect to the Jacobi L_2-norm is presented.

1. Introduction

Optimal *degree reduction* is one of the fundamental tasks in Computer Aided Geometric Design (CAGD) and therefore has attracted researchers' attention for several decades (Ait-Haddou, in press; Zhou and Wang, 2009; Ahn et al., 2004; Ahn, 2003; Kim and Ahn, 2000; Lutterkort et al., 1999; Watkins and Worsey, 1988). Used not only for data compression, CAD/CAM software typically requires algorithms capable of converting a curve (surface) of a high degree to a curve (surface) of a lower degree. Considering the problem coordinate-wise, the goal is formulated as follows: given a *univariate* polynomial p of degree n, find its best polynomial approximation q of degree m, $m < n$, with respect to a certain given norm.

The degree reduction can be seen as an inverse operation to the *degree elevation*. Whereas elevating polynomial degree from m to n is always possible, see e.g. (Hoschek and Lasser, 1993), because it is equivalent to expressing a polynomial $q \in \mathbb{P}_m$ in the basis of a larger linear space $\mathbb{P}_n$, $\mathbb{P}_m \subset \mathbb{P}_n$, the degree reduction is in general not. A natural alternative is then finding the best approximation that minimizes a certain error. This can be interpreted as projecting $p \in \mathbb{P}_n$ into $\mathbb{P}_m$. Depending on a particular norm defined on $\mathbb{P}_n$, various schemes for degree reduction were derived (Eck, 1993; Peters and Reif, 2000; Lee and Park, 1997; Kim and Moon, 1997; Brunnett et al., 1996; Ait-Haddou and Goldman, 2015).

An elegant resemblance between the L_2-norm and the Euclidean norm acting on the vector of Bernstein coefficients was revealed by Lutterkort et al. (1999). They proved that the least squares approximation of Bézier coefficients provides the best polynomial degree reduction in the L_2-norm. Two interesting generalizations of this result were achieved by Ahn et al. (2004) and by Ait-Haddou (submitted for publication). Ahn et al. (2004) showed that a weighted least squares approximation of Bézier coefficients provides the best *constrained* polynomial degree reduction in the L_2-norm. By constrained we understand that the original polynomial and its reduced-degree approximation match at the boundaries up to a specific

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continuity order. Ait-Haddou (submitted for publication) shows that the weighted least squares approximation of Bézier coefficients with Hahn weights provides the best polynomial degree reduction with respect to the Jacobi L_2-norm. In view of these two generalizations, it is natural to ask the following question:

(Q) – *Is there an analogue to the result of Lutterkort et al. (1999) for the constrained degree reduction with respect to the Jacobi L_2-norm?*

The Jacobi L_2-norm depends on two real parameters and a partial answer to the question (Q) is given in (Jianmin and Wang, 2003) for specific values of the parameters of the Jacobi L_2-norm.

In the present work, we give an affirmative answer to question (Q); namely we show that there exists a weighted inner product on the Bézier coefficients for which the problem of constrained degree reduction with respect to the Jacobi L_2-norm is equivalent to the problem of weighted least squares approximation of the Bézier coefficients.

Our methodology for answering question (Q) is very similar to Lutterkort et al. (1999) and its extension by Ahn et al. (2004). The main challenge lies in the construction of the adequate inner product of Bézier coefficients.

We note that a general solution to the problem of constrained degree reduction with respect to the Jacobi L_2-norm is derived in (Woźny and Lewanowicz, 2009). Although their solution does not require matrix inversion, the derivation is rather complicated because it requires an explicit computation of the dual bases of the discrete Bernstein bases. Moreover, their methodology does not involve the approach taken in (Lutterkort et al., 1999) and (Ahn et al., 2004). In contrast, our solution, even though it requires the computation of a single Moore–Penrose inverse, is simple and fits to the framework of (Lutterkort et al., 1999; Ahn et al., 2004).

The rest of the paper is organized as follows. In section 2, we prove that the best constrained polynomial degree reduction with respect to the Jacobi L_2-norm is equivalent to a weighted least squares approximation of Bézier coefficients with factored Hahn weights. We demonstrate how to compute the degree-reduced polynomials in Section 3, present several examples in Section 4, and finally conclude the paper in Section 5.

2. Constrained polynomial degree reduction with Jacobi norms

Denote by $\mathbb{P}_n$ the linear space of polynomials of degree at most n and let $\mathbf{B}^n$ be its Bernstein–Bézier (BB) basis and $\mathbf{Q}^n$ be its Lagrange basis with respect to the nodes $(0, 1, \ldots, n)$, i.e.,

$$\mathbf{B}^n := [B_0^n, \ldots, B_n^n], \quad \text{where} \quad B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad t \in [0,1], \quad \text{and}$$

$$\mathbf{Q}^n := [Q_0^n, \ldots, Q_n^n], \quad \text{where} \quad Q_i^n(t) = \prod_{j=0,\, j\neq i}^{n} \tfrac{t-j}{i-j}.$$

Let $\mathbb{P}_m$ be a subspace of $\mathbb{P}_n$, $m < n$ and let k and l be two non-negative integers such that $k + l \leq m + 1$, we define $\mathbb{P}_m^{k,l}$ as:

$$\mathbb{P}_m^{k,l} = \{f \in \mathbb{P}_m : f^{(i)}(0) = 0,\ i = 0, 1, \ldots, k-1;\ f^{(j)}(1) = 0,\ j = 0, \ldots, l-1\}.$$

That is, $\mathbb{P}_m^{k,l}$ is a linear space of polynomials of degree at most m with k vanishing derivatives at $t = 0$ and l vanishing derivatives at $t = 1$. Moreover, we define

$$\mathbb{Q}_m^{k,l} = \{f \in \mathbb{P}_m : f(i) = 0,\ i = 0, 1, \ldots, k-1 \text{ and } i = n-l+1, \ldots, n\}.$$

Let $\alpha > -1$ and $\beta > -1$ be two real numbers and define the Jacobi inner product in $\mathbb{P}_n$ by

$$< p, q >_{L_2} = \int_0^1 t^{\alpha}(1-t)^{\beta} p(t) q(t) \mathrm{d}t. \tag{1}$$

Considering the vectors of BB coefficients of p and q, $\mathbf{p} = [p_0, \ldots, p_n]^T$ and $\mathbf{q} = [q_0, \ldots, q_n]^T$, respectively, we define the following weighted Euclidean inner product of the BB coefficients

$$< p, q >_{E_2} = < \mathbf{B}^n \mathbf{p}, \mathbf{B}^n \mathbf{q} >_{E_2} = \sum_{i=k}^{n-l} w_i p_i q_i, \tag{2}$$

with the weights

$$w_i = \frac{\binom{n}{i}}{\binom{i}{i-k}\binom{n-i}{n-i-l}} (\alpha+1)_{k+i} (\beta+1)_{n-i+l}, \tag{3}$$

where $(a)_s = a(a+1)\ldots(a+s-1)$ denotes the Pochhammer symbol with the convention that $(a)_0 := 1$.