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Matrix weighted rational curves and surfaces ☆

Xunnian Yang

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

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ABSTRACT

Rational curves and surfaces are powerful tools for shape representation and geometric modeling. However, the real weights are generally difficult to choose except for a few special cases such as representing conics. This paper presents an extension of rational curves and surfaces by replacing the real weights with matrices. The matrix weighted rational curves and surfaces have the same structures as the traditional rational curves and surfaces but the matrix weights can be defined in geometric ways. In particular, the weight matrices for the extended rational Bézier, NURBS or the generalized subdivision curves and surfaces are computed using the normal vectors specified at the control points. Similar to the effects of control points, the specified normals can be used to control the curve or the surface's shape efficiently. It is also shown that matrix weighted NURBS curves and surfaces can pass through their control points, thus curve or surface reconstruction by the extended NURBS model no longer needs solving any large system but just choosing control points and control normals from the input data.

1. Introduction

In Computer Aided Geometric Design, rational curves and surfaces like

$$R(\xi) = \frac{\sum_i \omega_i P_i \phi_i(\xi)}{\sum_i \omega_i \phi_i(\xi)}, \quad \xi \in \Xi \tag{1}$$

where P_i are the control points, ω_i are the scalar weights and $\phi_i(\xi)$ are the blending functions defined on a 1D or 2D domain, are powerful tools for shape representation, modeling and reconstruction. Particularly, polynomial or rational Bézier curves and surfaces, NURBS (non-uniform rational B-spline) curves and surfaces and the generalized subdivision surfaces are widely used in CAD or computer animation industry (Farin, 2001; DeRose et al., 1998; Müller et al., 2006). As its compact representation of freeform as well as analytical curves and surfaces, NURBS has even become de facto industry standard in CAD commercial software (Piegl and Tiller, 1997).

Except for representing typical curves or surfaces such as conics or rotational surfaces, the potentials of rational curves and surfaces have not been explored thoroughly for geometric modeling. One possible reason is due to the limitations of the current representation. Generally, the geometric meaning of weights of rational curves and surfaces is not as clear as their control points. High order rational curves and surfaces are useful for high quality shape modeling (Cashman et al., 2009), but how to construct them is not as clear as desired. Rational curves and surfaces are also promising for shape reconstruction from scanned data but the determination of the weights is not a simple task (Xie et al., 2012). If some

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E-mail address: yxn@zju.edu.cn.

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differential quantities or sharp features should be considered, the reconstruction of curves and surfaces becomes nonlinear and optimization techniques have to be employed (Gofuku et al., 2009; Wu et al., 2013; Ma et al., 2015).

A geometric approach to construct rational curves and surfaces is the technique of dual Bézier curves and surfaces which treat the Bézier control points as control lines or control planes (Hoschek, 1983). Rational Bézier curves and surfaces are obtained as the envelopes of the dual Bézier curves or the dual Bézier surfaces. Even this is usually feasible, the denominator of an obtained rational Bézier curve or surface may vanish when the initial control lines or control planes have not been properly given. Another interesting extension of rational Bézier curves is complex rational Bézier curves (Sánchez-Reyes, 2009). Complex rational Bézier curves replace the scalar weights of the classical rational Bézier curves by complex numbers and several typical plane curves can be represented by complex rational Bézier curves of lower degrees. However, complex rational Bézier curves only lie in 2D space. Other interesting extensions of Bézier or NURBS curves and surfaces include plus curves and surfaces which use parameterized control lines (Goshtasby, 2005), curves and surfaces with basis functions that contain shape parameters (Juhász and Róth, 2013) or spline curves and surfaces with added geometric details controlled by vectors (Kosinka et al., 2015).

In this paper we present extended representation models of rational curves and surfaces with the following goals: (1) The new models maintain the same structures as the usual ones; (2) The new models are compatible with the traditional rational curves and surfaces; (3) The new weights permit geometric definitions and can be used for easy shape editing; (4) Curve or surface reconstruction using the extended rational models is simple and direct.

A rational curve or surface in $\mathbb{R}^d$ can be regarded as the perspective projection of a non-rational curve or surface in $\mathbb{R}^{d+1}$ (Farin, 1999). Similarly, we define rational curves and surfaces in $\mathbb{R}^d$ by the mapping of homogeneous curves or homogeneous surfaces in $\mathbb{R}^{(d+1)\times d}$. Concretely, we should only replace the scalar weights in Equation (1) with $d \times d$ matrices. If the denominator matrix function is non-singular over the whole parameter domain, a matrix weighted rational curve or surface will be obtained. It is clear that the extended rational curves and surfaces have the same structures as the traditional ones. If all the weight matrices are defined by multiplying a set of real numbers with the same non-singular matrix, the matrix weighted rational curve or surface will degenerate to a conventional rational curve or surface.

To achieve the third or even the fourth goal, we propose to compute the weight matrices based on normal vectors and real numbers specified at the control points of the curves or the surfaces. This is motivated by the observation that a rational curve or surface is in fact the solution to least-squares fitting to the control points with weights given by a set of basis functions. If we fit a curve or a surface to the points as well as to a set of lines or planes that pass through the points, we obtain a rational curve or a rational surface with matrix weights. These curves and surfaces can be controlled using control points together with control normals. Moreover, matrix weighted NURBS curves and surfaces can even pass through their control points, thus curve or surface reconstruction by the proposed models becomes easy because the control points of the reconstructed curve or surface should only be selected from the input data.

As the popular subdivision schemes proposed by Catmull and Clark (1978), Doo and Sabin (1978) or Loop (1987) are just the generalized B-spline surfaces with arbitrary topology control meshes, they can be extended to matrix weighted rational subdivision surfaces and the shapes of the extended subdivision surfaces can be controlled using control points and control normals. Similarly, other B-spline or NURBS compatible subdivision surfaces, for example the schemes in Sederberg et al. (1998), Stam (2001), Schaefer and Goldman (2009), can be generalized to matrix weighted rational subdivision surfaces too.

The paper is structured as follows. Section 2 presents the general definition and basic properties of matrix weighted rational curves and surfaces. More details about matrix weighted rational Bézier curves and matrix weighted NURBS curves are given in Section 3 and Section 4, respectively. Section 5 is devoted to matrix weighted rational parametric surfaces and matrix weighted rational subdivision surfaces. In Section 6 we present several interesting examples to show the potential applications of the proposed models. Section 7 concludes the paper.

2. Basics of matrix weighted rational curves and surfaces

In this section we present the definition of general matrix weighted rational curves and surfaces. A method for defining the weight matrices using specified normals and some basic properties of the obtained curves and surfaces will also be given.

2.1. Definition

Assume that $P_0, P_1, \ldots, P_n$ are a sequence or a set of points lying in $\mathbb{R}^d$ and $M_i \in \mathbb{R}^{d\times d}$, $i = 0, 1, \ldots, n$ are a set of known matrices. Suppose that $\phi_i(t) \geq 0$, $i = 0, 1, \ldots, n$ are a set of blending functions defined over a 1D or 2D domain Ξ. If the matrix function $M(\xi) = \sum_{i=0}^{n} M_i\phi_i(\xi)$ is non-singular for any $\xi \in \Xi$, a *valid* matrix weighted rational curve or surface is given by

$$Q(\xi) = \left[\sum_{i=0}^{n} M_i\phi_i(\xi)\right]^{-1} \sum_{i=0}^{n} M_i P_i\phi_i(\xi), \quad \xi \in \Xi \tag{2}$$

Particularly, if we choose $M_i = \omega_i W$, where $\omega_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$ and W is a $d \times d$ matrix that satisfies $|W| \neq 0$, the matrix weighted rational curve or surface $Q(\xi)$ will degenerate to a conventional rational curve or surface as given by Equation (1).