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Detection of critical points of multivariate piecewise polynomial systems ☆

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ABSTRACT

We propose a general scheme for detecting critical locations (of dimension zero) of piecewise polynomial multivariate equation systems. Our approach generalizes previously known methods for locating tangency events or self-intersections, in contexts such as surface–surface intersection (SSI) problems and the problem of tracing implicit plane curves. Given the algebraic constraints of the original problem, we formulate additional constraints, seeking locations where the differential matrix of the original problem has a non-maximal rank. This makes the method independent of a specific geometric application, as well as of dimensionality. Within the framework of subdivision based solvers, test results are demonstrated for non-linear systems with three and four unknowns.

1. Introduction and related work

The general problem of finding all solutions of algebraic equation systems in a given domain, arises in various contexts in Computer Aided Design (CAD), engineering, robotics, or whenever the geometry governing the problem can be mapped to (or represented by) a set of algebraic (non-linear in general) equations. The subdivision approach, exploiting properties of the Bernstein/B-spline basis functions, has been extensively investigated for several decades – introducing algorithms for finding roots of a univariate polynomial such as in Lane and Riesenfeld (1981), solutions of fully constrained multivariate systems (Sherbrooke and Patrikalakis, 1993; Elber and Kim, 2001 and more) as well as for under-constrained systems (Hanniel and Elber, 2007; Liang et al., 2008 and more). Typically, the generic methods used in advanced solvers to guarantee the topology of the solution set (number of roots, number of connected components, loops, closed surfaces, etc.) rely on some regularity (or transversality) assumptions which may slightly vary according to the application. However, topological guarantee near singular locations is treated separately, and is usually much more difficult to achieve (if at all).

Results and algorithms related to critical points detection are known, in various geometric contexts. In Grandine and Klein IV (1997), Hass et al. (2007), implicit planar curves which may admit self-intersections are traced. The self-intersections are first identified as critical points of the underlying implicit function, $f(x, y) = 0$, namely the solutions of the fully determined system: $\nabla f(x, y) = \bar{0} \in \mathbb{R}^2$. A numerical method with topological guarantee for implicit planar curves is given in Burr et al. (2008), which also detects isolated singularities and computes their degrees, using the number of connected components of certain topological structure in the neighborhood of the singularity.

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Critical points in the context of curve-surface intersection and surface–surface intersection are identified as tangency events of the two parametric geometries involved. For example, in Hu et al. (1997), tangency events of a parametric curve $r(t)$ and a parametric surface $R(u, v)$, are found by adding an orthogonality condition: $\langle r'(t), R_u(u, v) \times R_v(u, v)\rangle = 0$, to the original intersection requirement: $r(t) = R(u, v)$. In a similar manner, the additional conditions for tangency events of two parametric surfaces $R(u, v)$ and $P(s, t)$, are formulated to require that the two normal vectors at the intersection point are collinear.

Further, the dynamic version of the surface–surface intersection problem (Chen, 2008; Chen et al., 2007) is another application where the detection of critical points is essential: as the parametric surfaces evolve continuously with respect to a third parameter (time/control variable), the critical points are the events where intersection curve components may appear/disappear/merge and split. These are the topological events, where the solution curves may undergo topological changes, and they are characterized by tangency events of the evolving surfaces. In Chen (2008), Chen et al. (2007), they are recognized as the locations where a specific projection mapping is degenerate. However, no general method to locate the critical points of a smooth function $F : D \subset \mathbb{R}^n \to \mathbb{R}^k$, $k \leq n$ is provided.

Critical points are also used to determine the topology of an iso-surface of the form $f(x, y, z) = const$, using established results on the classification of critical points from Morse theory. In Stander and Hart (2005), the critical points are found as the solutions of $\nabla f(x, y, z) = \bar{0} \in \mathbb{R}^3$. Each such point corresponds to a critical value of f. The topological change in the iso-surfaces, as the function values vary through the critical one, is then identified using the Hessian value at the singular location. Consequently, a topologically correct triangular mesh is interactively updated. Such concepts are also used in Ni et al. (2004), where an appropriate Morse function is constructed on a polygonal mesh, such that its critical points can be optimized (their number can be controlled by the user). The critical points of the chosen function are then used to interrogate the topology of the mesh, and to separate it into disc-topology patches.

Other applications of critical point analysis arise in the context of bisector curves, offset curves and medial axis computation (Seong et al., 2010; Muthuganapathy et al., 2011; Johnson and Cohen, 2009; Musuvathy, 2011). These methods use critical point analysis of certain distance functions to locate transition events in the required solution manifolds. To the best of our knowledge, there's no (subdivision based) general method for finding the critical points of a smooth function $F : D \subset \mathbb{R}^n \to \mathbb{R}^k$, $k \leq n$. Although critical point analysis is widely used, it is usually of a specific function, intimately related to the problem or application domain. Our approach assumes no knowledge of the underlying motivation that gave rise to the equation solving problem or its dimensions. This generality, however, also has its drawbacks, as we discuss in further detail in Section 4.

The rest of this paper is organized as follows: In Section 2, the method for detecting critical points is detailed. Section 3 provides test results, for several types of problems and dimensions. Finally, Section 4 concludes and discusses future work.

2. Critical points detection

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^k$ $(k \leq n)$, be a (piecewise) polynomial, at least C^1 smooth, and given in a tensor product B-spline form. The vector valued function F is defined on the axis parallel, n dimensional compact box:

$$D = [a_1, b_1] \times \ldots \times [a_n, b_n],$$

which may formally be considered as a subset of an open set $U \subset \mathbb{R}^n$ on which F is defined and smooth. The scalar components of F are denoted by: $F = (f_1, \ldots, f_k)$. In this section, we describe a method for finding the critical points of:

$$F(\bar{x}) = \bar{0}, \tag{1}$$

using a (subdivision based) multivariate constraint solver. First, recall that the *differential* of F at $p \in D$, denoted dF_p, is the linear map, matrix representation of which has the partial derivatives of F evaluated at p as its elements:

$$[dF_p]_{ij} = \frac{\partial f_i}{\partial x_j}(p).$$

The critical points of F are generally defined by the following (Do Carmo, 1976):

Definition 1. Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^k$, defined in an open set U of $\mathbb{R}^n$, we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^k$ is not a surjective (onto) mapping. The image $F(p) \in \mathbb{R}^k$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^k$ which is not a critical value is called a *regular value* of F.

Remark 2. We are not interested, typically, in *all* the critical points of F, but only in those that are solution points as well. However, as will be evident shortly, the proposed method can be easily adopted to find all the critical points, rather than only those that are solution points (i.e. belonging to the critical value $\bar{0} \in \mathbb{R}^k$).

Prior to proceeding to solution details, the following clarification is in order. The singularities (or critical points) we seek are of the solution manifold, implicitly represented by the underlying equations system. This is not to be confused with the input equations, which may admit discontinuities of their own, but are trivial to handle: Since we assume a B-spline