Computer Aided Geometric Design 31 (2014) 631–641

Contents lists available at ScienceDirect

Computer Aided Geometric Design

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Using moving planes to implicitize rational surfaces generated from a planar curve and a space curve ☆

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ARTICLE INFO

Article history:
Received 20 March 2013
Received in revised form 8 October 2013
Accepted 19 January 2014
Available online 3 August 2014

Keywords:
Implicitization
μ-Basis
Moving planes
Rational surfaces

ABSTRACT

A rational surface

$$S(s,t)=\big(A(s)a(t),B(s)b(t),C(s)c(t),C(s)d(t)\big) \tag{1}$$

can be generated from a rational planer curve $\mathbf{P}^*(s)=(A(s),B(s),C(s))$ and a rational space curve $\mathbf{P}(t)=(a(t),b(t),c(t),d(t))$. Let $\mathbf{P}^*(s)$ pass through the point $(1,1,1)$. Then the surface $\mathbf{S}(s,t)$ goes through the space curve $\mathbf{P}(t)$. Moreover on each $z=z^*$-plane, the cross section of the surface $\mathbf{S}(s,t)$ is a stretching or shrinking of the planar curve $\mathbf{P}^*(s)$, such that the point $(1,1,z^*,1)$ travels to the point $\mathbf{P}(t)\cap\{z=z^*\}$. Using moving planes, we provide a new technique to implicitize this kind of rational surface. We find four moving planes that follow the surface from which we construct a sparse matrix whose size is just the degree of the surface with entries linear in x, y, z, w. We prove that the determinant of this matrix is the exact implicit equation of the surface $\mathbf{S}(s,t)$ without any extraneous factors. Examples are presented to illustrate our methods.

1. Introduction

We are going to investigate surfaces generated from a planar curve and a space curve. To explain our construction, we begin with the familiar surfaces of revolution.

A surface of revolution can be generated by a circle whose center moves up and down along an axis line and whose radius is controlled by the distance of points on a planar directrix $\mathbf{P}(t)$ from the axis – see Fig. 1. Many man made artifacts are surfaces of revolution, since surfaces of revolution are easy to design and control.

More generally, we find many other things, both man-made and natural, such as pyramids, square architectural columns and star fruit: these objects are surfaces with double planar directrices. The shapes of cross sections are now controlled by a planar curve $\mathbf{P}^*(s)$; the other planar directrix $\mathbf{P}(t)$ controls the size of the cross sections along the axis – see Fig. 2. In these two kinds of surfaces, each cross section in the x-direction and the y-direction are enlarged or contracted by the same amount. Thus the point $(1,1,0,1)$ travels to the point $(k^*,k^*,z^*,1)$ when $\mathbf{P}^*(s)$ travels along the z-axis, where $(k^*,z^*,1)$ is a point on the planer directrix $\mathbf{P}(t)$, see Figs. 1 and 2.

Now we want to design a new kind of surface: on each cross section we enlarge or contract the shape of the planar directrix $\mathbf{P}^*(s)$ along the x-direction and the y-direction but in different proportions. Thus the point $(1,1,0,1)$ travels to

☆ This paper has been recommended for acceptance by Falai Chen.
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http://dx.doi.org/10.1016/j.cagd.2014.01.001
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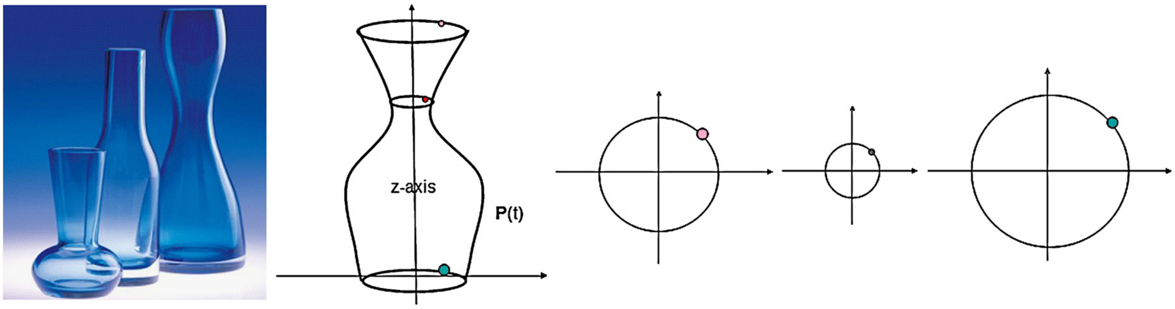

Fig. 1. A surface of revolution.

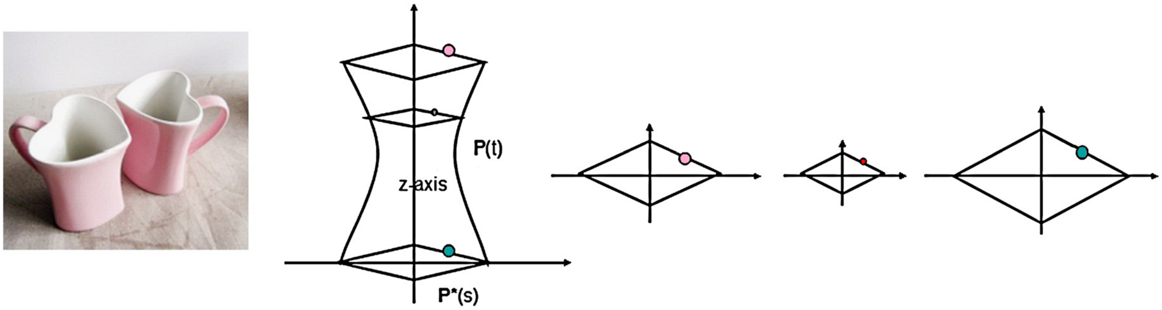

Fig. 2. A surface with double planar directrices.

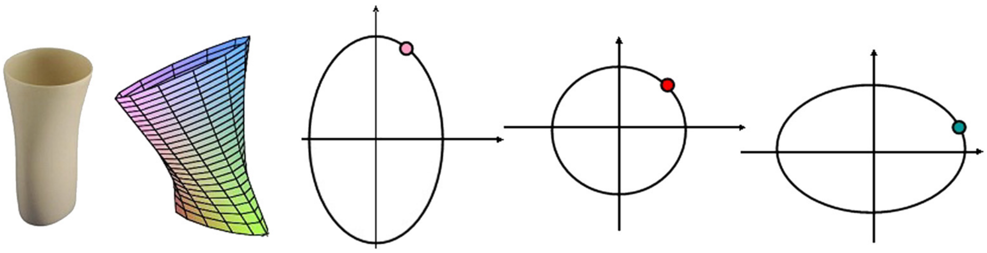

Fig. 3. A surface generated by a planar curve and a space curve.

$(x^*, y^*, z^*, 1)$ when $\mathbf{P}^*(s)$ travels along the z-axis, where $(x^*, y^*, z^*, 1)$ is always a point on the space curve $\mathbf{P}(t)$ – see Fig. 3 for an example. We can control the cross sections to become circles, long ellipses and short ellipses on the same surface.

While we can easily draw a surface using parametric equations, the implicit equation is also needed to represent and analyze the shape. The goal of this paper is to find the implicit equation of such surfaces from the rational parameterizations shown in Eq. (1). There are many generic methods for implicitizing rational curves and surfaces such as resultants, Grobner bases, and the method of moving lines and planes (Gonzalez-Vega et al., 2004; Kotsireas, 2004; Sederberg and Chen, 1995; Cox et al., 1998a). Here we shall develop an efficient technique with very little computation to implicitize a rational surface with a planar directrix and a space directrix by using moving planes.

Moving lines and moving planes were first introduced in Sederberg and Chen (1995). Moving lines are used to provide a compact representation for the implicit equation of a rational parametric curve (Cox et al., 1998b). Later, moving planes were applied to implicitize ruled surfaces (Chen et al., 2001), Steiner surfaces (Wang and Chen, 2012), surfaces with base points (Wang et al., 2008; Chen et al., 2005), surfaces of revolution (Shi and Goldman, 2012) and surfaces with a pair of planar directrices (Shi et al., 2012).

Here we shall show how to find four moving planes that follow the surfaces $\mathbf{S}(s,t)$ of bidegree (m,n) generated from a planar curve and a space curve. Using these four moving planes, we construct a $2mn \times 2mn$ Sylvester style matrix whose entries are linear in x, y, z, w. We prove that the determinant of this matrix is the exact implicit equation of the surface $\mathbf{S}(s,t)$ without any extraneous factors. In contrast, in the general resultant method for surfaces of bidegree (m,n), the Sylvester matrix is $6mn \times 6mn$. Moreover, compared to Grobner bases, our methods are much more efficient.

To derive these results, we proceed in the following fashion. We derive the parametric equations of the surface $\mathbf{S}(s,t)$ generated from a planar curve $\mathbf{P}^*(s)$ and a space curve $\mathbf{P}(t)$ in Section 2. Section 3 is a brief review of μ-bases for rational curves and moving planes for rational surfaces. In Section 4, we analyze the surface $\mathbf{S}(s,t)$ to get four moving planes that follow the surface. Using these four moving planes, we construct a Sylvester style matrix whose size is just the degree of