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Geometric Hermite interpolation by logarithmic arc splines ☆

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ABSTRACT

This paper considers the problem of G^1 curve interpolation using a special type of discrete logarithmic spirals. A "logarithmic arc spline" is defined as a set of smoothly connected circular arcs. The arcs of a logarithmic arc spline have equal angles and the curvatures of the arcs form a geometric sequence. Given two points together with two unit tangents at the points, interpolation of logarithmic arc splines with a user specified winding angle is formulated into finding the positive solutions to a vector equation. A practical algorithm is developed for computing the solutions and construction of interpolating logarithmic arc splines. Compared to known methods for logarithmic spiral interpolation, the proposed method has the advantages of unbounded winding angles, simple offsets and NURBS representation.

1. Introduction

Spirals, which have monotone curvatures, find wide applications in the fields of fair shape modeling, highway route design or artistic pattern design, etc. (Meek and Walton, 1992; Wang et al., 2004; Xu and Mould, 2009; Meek et al., 2012). Particularly, the clothoid spiral (also known as the Euler spiral) whose curvature is a linear function of its arc length, has often been used as a primary tool for curve completion or fair shape modeling (Kimia et al., 2003; Zhou et al., 2012). Another popular spiral is the logarithmic spiral whose radius of curvature is a linear function of its arc length. The study of logarithmic spirals goes back to Descartes and Jacob Bernoulli (Davis, 1993). Logarithmic spirals have many elegant properties and can be used to model fair shapes as well as natural objects (Harary and Tal, 2011).

As a generalization of Euler spirals and logarithmic spirals, Miura (2006) proposed a general equation for log-aesthetic curves. By choosing different values for a particular parameter, one can define various spirals by the equation. Except for a few special cases like circles, evaluation of log-aesthetic curves depends on numerical integration or computation of special functions (Ziatdinov et al., 2012a). If boundary points and tangents are given first, parameters for an interpolating spiral are usually determined by solving nonlinear systems or by searching strategies (Coope, 1992; Miura, 2000; Yoshida and Saito, 2006; Ziatdinov et al., 2012b).

Inspired by the fact that log-aesthetic curves are usually computed numerically or approximated by other types of curves such as polynomials or rational polynomials, one can construct interpolating spirals discretely or using polynomials directly (Baumgarten and Farin, 1997; Yoshida and Saito, 2009; Walton and Meek, 2013; Yoshida et al., 2013). Polynomials or rational polynomials that have approximate linear plots of log curvatures are quasi-log-aesthetic spirals. These spirals can be evaluated explicitly. However, these curves are not log-aesthetic spirals exactly and many quasi-log-aesthetic spirals have to be pieced together when a high accuracy of approximation is desired. Geometric Hermite interpolating curves with

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minimal energy can generate fair shapes (Yong and Cheng, 2004), but the Euler spiral and the logarithmic spiral are of special interest in shape modeling.

In this paper we consider G^1 Hermite interpolation by logarithmic arc splines. Our proposed algorithm is motivated by the equiangular property of logarithmic spirals and the high accuracy approximation of spirals by arc splines (Meek and Walton, 1999). By assuming that a logarithmic spiral is approximated by a sequence of smoothly connected circular arcs of equal angles and the curvatures or radii of curvatures of all arcs form a geometric sequence, we obtain a logarithmic arc spline. G^1 Hermite interpolation by logarithmic arc splines can be formulated as solving the free parameters from a simple equation. All solutions to the equation can be obtained using an efficient numerical method. As opposed to previous approaches which assumed bounded winding angles and unique interpolating curves, we have no such restrictions and all interpolating curves to the given boundary data can be obtained efficiently. As offsets of circular arcs are circular arcs, the offsets of logarithmic arc splines are easy to compute. Logarithmic arc splines and their offsets can be represented by NURBS or transformed into curvature continuous curves conveniently (Yang, 2004). Therefore, the proposed curve can be used as an efficient tool for shape modeling and CNC machining.

The paper is structured as follows. Section 2 briefly reviews important properties of logarithmic spirals and proposes a definition of a logarithmic arc spline. Section 3 describes basic formulations of G^1 curve interpolation by logarithmic arc splines. Theoretical analysis on the existence and algorithm steps for logarithmic arc spline interpolation are also presented. Several interesting examples are provided in Section 4, and they demonstrate the applicability of the proposed algorithm. Section 5 concludes the paper.

2. Logarithmic spiral and logarithmic arc spline

2.1. Basics of logarithmic spirals

A logarithmic spiral of which the pole lies at the origin can be represented by polar coordinates as

$$r(t) = r_0 e^{\lambda t}, \quad r_0 \in \mathbb{R}^+, \ \lambda \in \mathbb{R} \tag{1}$$

or, by Cartesian coordinates as

$$\mathbf{S}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 e^{\lambda t} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}. \tag{2}$$

Particularly, $\mathbf{S}(t)$ will approach the pole when λt approaches $-\infty$.

A logarithmic spiral arc can be defined by either of the above equations when the parameter t belongs to an interval $[t_1, t_2]$. The winding angle of the logarithmic spiral arc is obtained as $\phi = t_2 - t_1$ when $\lambda > 0$ or $\phi = t_1 - t_2$ when $\lambda < 0$. If the winding angle satisfies $|\phi| \le 2\pi$, the spiral arc is also referred as a *single-winding logarithmic spiral*; otherwise, it is a *multi-winding logarithmic spiral*.

Logarithmic spiral has several distinguished properties which make it a powerful tool for shape modeling. The clear or easily proved properties are listed with no proof.

Property 2.1. *The angle between any radial line and the tangent line that passes through the same point does not change when the point moves along the logarithmic spiral.*

This property is also known as the equiangular property, which was first observed by Rene Descartes. In particular, the angle φ between the radial line and the tangent line is computed by $\lambda = \cot\varphi$, where λ is the parameter as in Eq. (1).

Property 2.2. *Let $\mathbf{S}(t)$ be a logarithmic spiral, $k \in \mathbb{Z}^+$, the tangents at points $\mathbf{S}(t)$ or $\mathbf{S}(t + 2k\pi)$ are parallel and the angle between the tangent direction and the chord $\mathbf{S}(t + 2k\pi) - \mathbf{S}(t)$ is acute.*

Property 2.3. *Let $\mathbf{P}_a$ and $\mathbf{P}_b$ be the endpoints of a logarithmic spiral arc, the curvature decreasing from $\mathbf{P}_a$ to $\mathbf{P}_b$ and the winding angle being less than 2π. Assume α and β are the unsigned angles between $\mathbf{P}_b - \mathbf{P}_a$ and the tangent to the arc at $\mathbf{P}_a$ or between $\mathbf{P}_b - \mathbf{P}_a$ and the tangent to the arc at $\mathbf{P}_b$, respectively. It follows that $\alpha > \beta$.*

Proof. Without loss of generality we assume the logarithmic spiral is represented by Eq. (2) with $\lambda > 0$, and the endpoints of a logarithmic spiral arc are $\mathbf{P}_a = \mathbf{S}(t)$ and $\mathbf{P}_b = \mathbf{S}(t + \tau)$. If the winding angle τ is less than π, the logarithmic spiral arc is convex and short and the property holds based on Vogt's theorem (Theorem 3.17 in Guggenheimer, 1977).

We prove $\alpha > \beta$ for $\pi \le \tau < 2\pi$. Since $0 < \alpha, \beta < \pi$, we should only prove $\cos\alpha < \cos\beta$. From Eq. (2), we have

$$\cos\alpha = \frac{\mathbf{S}(t+\tau) - \mathbf{S}(t)}{\|\mathbf{S}(t+\tau) - \mathbf{S}(t)\|} \cdot \frac{\mathbf{S}'(t)}{\|\mathbf{S}'(t)\|} = \frac{e^{\lambda\tau}(\lambda\cos\tau + \sin\tau) - \lambda}{\sqrt{1 + e^{2\lambda\tau} - 2e^{\lambda\tau}\cos\tau}\sqrt{1+\lambda^2}}$$

and