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Differential geometry of non-transversal intersection curves of three parametric hypersurfaces in Euclidean 4-space ☆

Osmar Aléssio [a], Mustafa Düldül [b,*], Bahar Uyar Düldül [c], Sayed Abdel-Naeim Badr [d], Nassar H. Abdel-All [d]

[a] UFTM – Triângulo Mineiro Federal University, Institute of Exact Sciences, Natural and Education, Department of Mathematics, Uberaba, MG, Brazil
[b] Yıldız Technical University, Science and Arts Faculty, Department of Mathematics, Istanbul, Turkey
[c] Yıldız Technical University, Education Faculty, Department of Mathematics Education, Istanbul, Turkey
[d] Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

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ABSTRACT

The purpose of this paper is to present algorithms for computing all the differential geometry properties of non-transversal intersection curves of three parametric hypersurfaces in Euclidean 4-space. For transversal intersections, the tangential direction at an intersection point can be computed by the extension of the vector product of the normal vectors of three hypersurfaces. However, when the three normal vectors are not linearly independent, the tangent direction cannot be determined by this method. If normal vectors of hypersurfaces are parallel ($\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3$) we have tangential intersection, and if normal vectors of hypersurfaces are not parallel but are linearly dependent we have "almost tangential" intersection. In each case, we obtain unit tangent vector ($\mathbf{t}$), principal normal vector ($\mathbf{n}$), binormal vectors ($\mathbf{b}_1$, $\mathbf{b}_2$) and curvatures (k_1, k_2, k_3) of the intersection curve.

1. Introduction

In this paper, we worked with three parametric hypersurfaces in $\mathbb{R}^4$ and study the differential geometry properties of their intersection curves. The intersection of the hypersurfaces can be transversal (the normal vectors are linearly independent) and non-transversal (the normal vectors are linearly dependent). In the case of transversal intersection, the tangential direction at an intersection point can be computed simply by vector product of the normal vectors of the hypersurfaces. For that reason, various methods and techniques have been given for computing the differential geometry properties of transversal intersections of surfaces in $\mathbb{R}^3$, and of hypersurfaces in $\mathbb{R}^4$. However, when the normal vectors are linearly dependent at an intersection point, tangent direction cannot be determined by this method. Because of this, there exists little literature for computing the Frenet apparatus of the non-transversal intersection curves.

The geometric properties formulas for parametrically defined curves are well known in the classical literature on differential geometry in $\mathbb{R}^3$ (do Carmo, 1976; Spivak, 1975; Stoker, 1969; Struik, 1950; Willmore, 1959) and in the contemporary literature on geometric modeling (Farin, 2002; Hoschek and Lasser, 1993; Patrikalakis and Maekawa, 2002). The higher

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* Corresponding author.
E-mail addresses: osmar@matematica.uftm.edu.br (O. Aléssio), mduldul@yildiz.edu.tr (M. Düldül), buduldul@yildiz.edu.tr (B. Uyar Düldül), sayed_badr@ymail.com (S.A.-N. Badr), nhabdeal2002@yahoo.com (N.H. Abdel-All).

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curvatures of curves in Euclidean space can be found in textbooks such as Guggenheimer (1963), Klingenberg (1978), Kühnel (2006), Spivak (1999) and papers such as Esin and Hacısalihoğlu (1986), Gluck (1966).

The geometric properties formulas of implicitly defined curves of the **transversal intersection** can be found in the papers: Willmore describes how to obtain the unit tangent **t**, the unit principal normal **n**, and the unit binormal **b**, as well as the curvature k and the torsion τ of the intersection curve of two implicit surfaces in $\mathbb{R}^3$, using the operator $\triangle = \lambda \frac{d}{ds} = (h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z})$, where $h = \nabla f \times \nabla g$ (Willmore, 1959). Using the idea of Willmore, Düldül and Akbaba introduce some new operators for finding the Frenet apparatus of the transversal intersection curves of two (three) surfaces (hypersurfaces) in which at least one (hyper)surface is given parametrically (Düldül and Akbaba, 2014). Faux and Pratt give the curvature of an intersection curve of two parametric surfaces (Faux and Pratt, 1981). Hartmann provides formulas for computing the curvature k and geodesic curvature κ_g of the intersection curves for all three types of intersection problems in $\mathbb{R}^3$ (parametric–parametric, implicit–implicit and parametric–implicit), using the Implicit Function Theorem (Hartmann, 1996). Ye and Maekawa provide **t**, **n**, **b**, κ, τ using the vector α'' as linear combinations of the normal vectors of the surfaces and α''' as linear combination of the tangent vector and normal vectors of the surfaces. They also provide an algorithm for the evaluation of higher-order derivatives for transversal as well as tangential intersections for all three types of intersection problems in $\mathbb{R}^3$ (Ye and Maekawa, 1999). Goldman provides closed formulas for computing the curvature (k) and torsion (τ) of the intersection curve of two implicit surfaces in $\mathbb{R}^3$ and the curvature (κ) of the intersection curve in $(n+1)$ dimensions (Goldman, 2005). Aléssio provides curvature (k) and torsion (τ) for the transversal intersection of two implicit surfaces, using the Implicit Function Theorem (Aléssio, 2006). Aléssio and Guadalupe present formulas for curvature, geodesic torsion and geodesic curvature for the intersection curve of two spacelike surfaces in the Lorentz–Minkowski 3-space (Aléssio and Guadalupe, 2007). Aléssio derives unit tangent vector (t), normal vector (n), binormal vectors (b_1, b_2) and curvatures (κ_1, κ_2, κ_3) for the transversal intersections, using the Implicit Function Theorem and the method of X. Ye and T. Maekawa for 4 dimensions (Aléssio, 2009). Düldül provides a method for computing the Frenet vectors and the curvatures of the transversal intersection curve of three parametric hypersurfaces in four-dimensional Euclidean space (Düldül, 2010). Uyar Düldül and Düldül obtain the Frenet apparatus of the transversal intersection of three implicit hypersurfaces by extending the Willmore's method into 4-space (Uyar Düldül and Düldül, 2012). Abdel-All et al. provide a method for computing the Frenet vectors and the curvatures of the transversal intersection curve of implicit–parametric–parametric and implicit–implicit–parametric hypersurfaces in four-dimensional Euclidean space (Abdel-All et al., 2012a). Aléssio obtained the normal curvature ($\kappa_n^{f_i}$), the first geodesic curvature ($\kappa_{1g}^{f_i}$) and the first geodesic torsion ($\tau_{1g}^{f_i}$) for the transversal intersection curve of $n-1$ implicit hypersurfaces in $\mathbb{R}^n$ (Aléssio, 2012).

The geometric properties formulas of implicitly defined curves of **non-transversal intersection** can be found in the following papers: Kruppa describes that the tangential direction of the intersection curve at a tangential intersection point corresponds to the direction from the intersection point towards the intersection of the Dupin indicatrices of the two surfaces (Kruppa, 1957). Cheng (1989) and Markot and Magedson (1989, 1991) give solutions for parametric surfaces at isolated tangential intersection points, based on the analysis of the plane vector field function defined by the gradient of an oriented distance function of one surface from the other. Ye and Maekawa provide an algorithm for the evaluation of higher-order derivatives for tangential intersections for all three types of intersection problems in $\mathbb{R}^3$ (Ye and Maekawa, 1999). Abdel-All et al. provide an algorithm for the evaluation of geometry properties for tangential intersections of two implicit surfaces in $\mathbb{R}^3$ using the implicit function theorem (Abdel-All et al., 2012b).

In Section 2, we introduce some notations and reviews of the differential geometry of curves and hypersurfaces. In Section 3, we give the formulas which compute the properties of the non-transversal intersection curve of three parametric hypersurfaces. Finally, as an application of our methods, we give several examples of non-transversal intersection.

2. Preliminaries

Definition 2.1. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of four dimensional Euclidean space E^4. The *ternary product* (or *vector product*) of the vectors $\mathbf{x} = \sum_{i=1}^4 x_i \mathbf{e_i}$, $\mathbf{y} = \sum_{i=1}^4 y_i \mathbf{e_i}$, and $\mathbf{z} = \sum_{i=1}^4 z_i \mathbf{e_i}$ is defined by (Hollasch, 1991; Williams and Stein, 1964)

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

The ternary product $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ yields a vector that is orthogonal to **x**, **y**, and **z**.

Let $M \subset E^4$ be a regular hypersurface given by $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ and $\alpha : I \subset \mathbb{R} \to M$ be an arbitrary curve with arc-length parametrization. If $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by (Gluck, 1966)

$$\mathbf{t}' = k_1 \mathbf{n}, \qquad \mathbf{n}' = -k_1 \mathbf{t} + k_2 \mathbf{b}_1, \qquad \mathbf{b}_1' = -k_2 \mathbf{n} + k_3 \mathbf{b}_2, \qquad \mathbf{b}_2' = -k_3 \mathbf{b}_1, \tag{2.1}$$

where **t**, **n**, $\mathbf{b}_1$, and $\mathbf{b}_2$ denote the tangent, the principal normal, the first binormal, and the second binormal vector fields; k_i $(i = 1, 2, 3)$ are the ith curvature functions of the curve α. Also, since M is regular, the partial derivatives $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$