Computer Aided Geometric Design 31 (2014) 211–226

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Optimal arc spline approximation ☆

Georg Maier

FORWISS, University of Passau, Innstr. 43, 94032 Passau, Germany

ARTICLE INFO

Article history:
Received 19 April 2013
Received in revised form 16 December 2013
Accepted 27 February 2014
Available online 4 April 2014

Keywords:
Approximation
Arc spline
Biarc
SMAP

ABSTRACT

We present a method for approximating a point sequence of input points by a G^1-continuous (smooth) arc spline with the minimum number of segments while not exceeding a user-specified tolerance. Arc splines are curves composed of circular arcs and line segments (shortly: segments). For controlling the tolerance we follow a geometric approach: We consider a simple closed polygon P and two disjoint edges designated as the start s and the destination d. Then we compute a *SMAP (smooth minimum arc path)*, i.e. a smooth arc spline running from s to d in P with the minimally possible number of segments. In this paper we focus on the mathematical characterization of possible solutions that enables a constructive approach leading to an efficient algorithm.

In contrast to the existing approaches, we do not restrict the breakpoints of the arc spline to a predefined set of points but choose them automatically. This has a considerably positive effect on the resulting number of segments.

1. Introduction

Various applications require the approximation of planar point sequences by a suitable class of curves. Arc splines are curves composed of circular arcs and line segments (shortly: segments) which therefore have advantageous properties: exact offset and arc length computation, parameter free description and simple closest point calculation (cf. Maier, 2010). In particular, smooth, i.e. G^1-continuous arc splines have been studied in literature in the past few years. Ones of the first publications were Sandel (1937) and Sabin (1977). Arc splines were then further popularized in the publications (Meek and Walton, 1992, 1995). Often, methods are considered which guarantee that the resulting arc spline has a possibly small number of arcs and stays within a user-specified tolerance from the input data (cf. Piegl and Tiller, 2002; Heimlich and Held, 2008; Drysdale et al., 2008). For this purpose, corresponding algorithms mostly follow geometric approaches: In order to approximate a simple polygonal curve, the ε-tolerance zone is focused, which is a closed polygon deduced by the set of points which have an Euclidean distance of at most ε from the input points. In Heimlich and Held (2008) an efficient method is introduced which generates tolerance zones using Voronoi diagrams.

Biarcs, i.e. G^1 curve segments consisting of two circular arcs, have been used in a large number of algorithms for approximation or interpolation of given point and possibly tangent data. For an overview we refer to Heimlich and Held (2008). Usually, biarc algorithms fit biarcs or single arcs between selected pairs of points and use the connecting line segments of each two corresponding points for the tangent data. Doubtlessly, they are appropriate for various applications. However, there are cases where they suffer from the unnatural requirement that breakpoints and tangent data of the approximating arc spline have to be chosen from a given finite point set in advance. This is even more the case for the second main

☆ This paper has been recommended for acceptance by Ralph Martin.

E-mail address: georg.maier@uni-passau.de.

http://dx.doi.org/10.1016/j.cagd.2014.02.011

approach in literature which is based on interpolation (e.g. Yang, 2002). Note that restricting the loci of the breakpoints can have considerable influence on the resulting segment number: In Maier and Pisinger (2013), which deals with the approximation of a cyclic polygonal curve by G^0-continuous arc splines, one can find an illustrative example: For each biarc and interpolation based algorithm A and $n \in \mathbb{N}$, an open polygon K with n vertices can be created and an ε can be chosen such that A needs $O(n)$ segments but the SMAP algorithm needs only one segment for approximating K with a maximum tolerance of ε.

Thus, we are interested in an arc spline approximation achieving the minimal number of segments subject to the following: The solution is G^1-continuous and it stays inside a user-specified tolerance. Such a solution is called *Smooth Minimum Arc Path (SMAP)*.

The paper is organized as follows: In Section 2 we fix some basic definitions and notation needed in the main part, which deals with the two main tools used in this work: *alternating sequences* and *feasible direction sets* (cf. Sections 3 and 4). After the characterization of the so-called *visibility set* in Section 5 and the continuity properties of feasible directions in Section 6, we focus on n-visibility sets for $n > 1$ and finally the computation of a SMAP (cf. Sections 7 and 8). Finally, we address practical applications of the SMAP algorithm, and discuss possible extensions of the proposed method.

An extensive treatise of this topic including all special cases and more detailed proofs can be found in Maier (2010). In contrast, this paper is of a comprehensive nature and is more comfortable to read: We made some assumptions which allow a simplification of the notation and the proofs, which is still general enough for most practical applications.

2. Fundamentals

2.1. Basic definitions

For a set $A \subset \mathbb{R}^2$, ∂A denotes the boundary of A, $\bar{A}$ its closure and A° its interior. The set $\mathfrak{K}(A)$ is used for the set of all non-empty compact subsets of A. By

$$\mathrm{dist}(A, B) := \inf_{x\in A} \inf_{y\in B} \|x - y\|$$

we denote the *Euclidean distance* and use the abbreviation $\mathrm{dist}(x, A) := \mathrm{dist}(\{x\}, A)$. Likewise, the *Hausdorff metric* is defined as follows:

$$\mathfrak{h}(A, B) := \max\left\{\sup_{x\in A} \mathrm{dist}(x, B), \sup_{x\in B} \mathrm{dist}(x, A)\right\}.$$

We will deal with planar curves which are simple, i.e. curves without any self-intersections. The image set, also called *trace*, of a curve ω is denoted by $\mathrm{tr}(\omega) \subset \mathbb{R}^2$. To simplify the notation we often identify ω with $\mathrm{tr}(\omega)$. The order on $\mathrm{tr}(\omega)$ induced by the orientation of ω is denoted by $\prec_\omega$ and $[x, y]_\omega \subset \omega$ is the subset 'between' $x \in \omega$ and $y \in \omega$. For the tangent unit vector, if it exists, at a point $a \in \omega$, we use the symbol $\tau_\omega(a)$.

Definition 1. A simple curve Γ in $\mathbb{R}^2$ is called *arc spline* if its trace is a union $\Gamma = \bigcup_{i=1}^n A_i$ of finitely many arcs or line segments. The minimally possible number $n \in \mathbb{N}$ for the above condition is called *segment number of* Γ. We use the abbreviation $|\Gamma| := n$. Arc splines γ with $|\gamma| = 1$ are simply called *segments*. Every segment γ has exactly one *starting point* $s(\gamma)$ and one *endpoint* $e(\gamma)$. An arc spline Γ is called *smooth* if it is G^1-continuous, i.e. $\tau_\Gamma(x^+) = \tau_\Gamma(x^-)$ for all $x \in \Gamma$.

Let x and a be two distinct points in the plane and let v be a direction, i.e. a point of the unit sphere $\mathbb{S}^1$. If $v \neq \frac{x-a}{\|x-a\|}$, then there exists exactly one segment $\gamma := \gamma_{x,a,v}$ with $s(\gamma) = x$, $e(\gamma) = a$ and $\tau_\gamma(a) = v$.

2.2. Tolerance channels and circular visibility

The proposed approach deals with an approximation up to a given tolerance, which can possibly vary locally. As described above we control the approximation error by considering only smooth arc splines staying inside the so-called tolerance channel, which is a closed polygon deduced from the input points and two designated edges. The width of this channel represents the user-specified maximum tolerance, which can vary locally as well. The canonical shape of a tolerance channel modeling a maximum error ε is given by the set of points which have an Euclidean distance to the open polygon running through the set of input points P of at most ε. Generally speaking, this ε-offset is a region formed from strips of width 2ε which are centered at the polygon edges. Thus, in a neighborhood of sharp corners this doesn't guarantee that the curve remains close to the given points. In Heimlich and Held (2008) however a method is introduced which avoids this problem and computes such tolerance channel in an efficient manner. Therefore, we do not deal with the generation of tolerance channels for a given point set and user-specified tolerance within this scope.