Computer Aided Geometric Design 31 (2014) 245–257

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Optimizing at the end-points the Akima's interpolation method of smooth curve fitting ☆

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ARTICLE INFO

Article history:
Received 14 June 2013
Received in revised form 30 January 2014
Accepted 14 March 2014
Available online 1 April 2014

Keywords:
Cubic splines of Hermite type
The Akima's interpolation procedure
Minimal deviation from the polygonal line
Partial quadratic oscillation in average

ABSTRACT

In this paper we propose an optimized version, at the end-points, of the Akima's interpolation method for experimental data fitting. Comparing with the Akima's procedure, the error estimate, in terms of the modulus of continuity, is improved. Similarly, we optimize at the end points the Catmull–Rom's cubic spline. The properties of the obtained splines are illustrated on a numerical experiment.

1. Introduction

In the problem of smooth curve fitting of experimental data, a remarkable result was obtained by Akima (1970). The Akima's interpolation method provides a natural and more suitable procedure for the smooth fitting of the data (x_i, y_i), $i = \overline{0,n}$, with $y_i \in \mathbb{R}$, $\forall i = \overline{0,n}$, and x_i, $i = \overline{0,n}$, being the knots of a grid, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. This method is based on the two-point Hermite's interpolation cubic polynomial

$$s(x) = \frac{(x_i - x)^2[2(x - x_{i-1}) + h_i]}{h_i^3} \cdot y_{i-1} + \frac{(x - x_{i-1})^2[2(x_i - x) + h_i]}{h_i^3} \cdot y_i + \frac{(x_i - x)^2(x - x_{i-1})}{h_i^2} \cdot m_{i-1} - \frac{(x - x_{i-1})^2(x_i - x)}{h_i^2} \cdot m_i, \quad x \in [x_{i-1}, x_i],\ i = \overline{1,n} \tag{1}$$

where $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$, and in the case that $y_i = f(x_i)$, $i = \overline{0,n}$, are the values of a function $f \in C^1[a,b]$ on the given knots, the values m_i, $i = \overline{0,n}$, stands for the derivatives $f'(x_i)$. When the values $f'(x_i)$ of the derivatives are known, in the previous formula (1) we can put, $m_i = f'(x_i)$, $i = \overline{0,n}$, but elsewhere the values m_i, $i = \overline{0,n}$, have to be determined. In the Akima's method, these values are computed using a local procedure based on geometric reasons. More exactly, for five given points $M_i(x_i, y_i)$, $i = \overline{1,5}$, are computed the slopes $p_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $i = \overline{1,4}$, and then, starting from a proportion that uses some of the obtained segments, is suggested the following value for the tangent in the point $M_2(x_2, y_2)$:

$$m_2 = \frac{|p_4 - p_3| \cdot p_2 + |p_2 - p_1| \cdot p_3}{|p_4 - p_3| + |p_2 - p_1|}. \tag{2}$$

☆ This paper has been recommended for acceptance by T. Lyche.

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http://dx.doi.org/10.1016/j.cagd.2014.03.001
0167-8396/

This formula (2) is generalized considering the slopes $p_i = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$, $i = \overline{0, n-1}$, together with the derivatives

$$m_i = \frac{|p_{i+1} - p_i| \cdot p_{i-1} + |p_{i-1} - p_{i-2}| \cdot p_i}{|p_{i+1} - p_i| + |p_{i-1} - p_{i-2}|}, \quad i = \overline{2, n-2}. \tag{3}$$

In order to extend the formula (3) for $i = \overline{0, n}$, the previously computed slopes are not enough and therefore, Akima proposes the construction of four new supplementary slopes p_{-1}, p_{-2}, p_n, p_{n+1}, based on a reasoning in the framework of a particular case (equidistant grid and exactness for second order polynomials on the end intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$):

$$p_{-1} = 2p_0 - p_1, \qquad p_{-2} = 3p_0 - 2p_1, \qquad p_n = 2p_{n-1} - p_{n-2}, \qquad p_{n+1} = 3p_{n-1} - 2p_{n-2}. \tag{4}$$

Since the artificial introduction of the four slopes well performs only in the particular case of equidistant grids, this is not a strong point of the Akima's method. For this reason we propose here an optimal procedure for the computation of the left unspecified derivatives m_0, m_1, m_{n-1}, m_n (the other derivatives m_i, $i = \overline{2, n-2}$, are computed using (3)). In a recent work (see Bica, 2012), we have defined a special functional, the quadratic oscillation in average (for this notion, see Bica, 2012), and we have computed the values m_i, $i = \overline{0, n}$, in order to minimize the quadratic oscillation in average.

Generally, for given points (x_i, y_i), $i = \overline{0, n}$, the formula (1) leads to a cubic spline $s \in C^1[a, b]$ for any values of m_i, $i = \overline{0, n}$. There are proposed many procedures to estimate the values m_i, $i = \overline{0, n}$. One of them is to require $s \in C^2[a, b]$ and to impose two additional conditions, for instance natural end conditions $s''(a) = s''(b) = 0$ (see Ahlberg et al., 1967 and Micula and Micula, 1999), obtaining a tridiagonal linear system for the unknown m_i, $i = \overline{0, n}$, diagonally dominant, which is exactly solved using an iterative algorithm (see Ahlberg et al., 1967, pages 14–15) and resulting the Hermite type natural cubic spline. For the error in the approximation of the differences $|f'(x_i) - m_i|$, $i = \overline{0, n}$, for this natural cubic spline (and for complete and periodic cubic splines $s \in C^2[a, b]$) see Kershaw (1972).

For cubic splines $s \in C^1[a, b]$, there are some ways to compute empirically the values m_i, $i = \overline{0, n}$. One of the simplest procedure is to consider the three-point finite difference

$$m_i = \frac{y_{i+1} - y_i}{2(x_{i+1} - x_i)} + \frac{y_i - y_{i-1}}{2(x_i - x_{i-1})}, \quad i = \overline{1, n-1}$$

and one-sided difference for the end-points, $m_0 = \frac{y_1-y_0}{x_1-x_0}$, $m_n = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}$. Another ways generated by geometric reasons lead to the Kochanek–Bartels splines (see Knott, 2000 and Kochanek and Bartels, 1984) characterized by three control parameters (tension, bias, and continuity), and to cardinal splines with $m_i = (1 - c) \cdot \frac{y_{i+1}-y_{i-1}}{x_{i+1}-x_{i-1}}$, $i = \overline{1, n-1}$, and $c \in [0, 1]$ be a tension parameter. Two remarkable particular cases of cardinal splines are the following: the Catmull–Rom spline (for $c = 0$) and the cubic spline with zero tangents (when $c = 1$, the maximal value of the tension parameter). In Reimer (1984), the best error bound is obtained in terms of the modulus of continuity for periodic cardinal splines. In Ichida et al. (1976), for the case of experimental data y_i, $i = \overline{0, n}$, affected by measurement errors, both the values y_i, $i = \overline{0, n}$, and m_i, $i = \overline{0, n}$, are computed by the use of an algorithm of least squares fitting minimizing an appropriate residual.

Another way to compute the values m_i, $i = \overline{0, n}$, in the Hermite type cubic splines, is to require the preservation of some graphic properties (see Burmeister et al. (1985) for convex splines with minimal L^2-norm of the second derivative, Wolberg and Alfy (2002) for monotonic splines with minimal curvature-type strain energy of the curve, Dietze and Schmidt (1988) for shape-preserving cubic splines with minimal curvature, and Conti et al. (1996) for shape preserving C^1-Hermite type cubic splines), or to impose the minimization of some functionals like geometric curvature (see Burmeister et al., 1985; Dietze and Schmidt, 1988; Micula and Micula, 1999), or like energy-type curvature (see Wolberg and Alfy, 2002), or minimizing the L^2-norms of all spline derivatives $s^{(r)}$, $r = 0, 1, 2, 3$ (see Kobza, 2002). In Yong and Cheng (2004), optimized geometric Hermite curves with minimum strain energy is constructed (both in $\mathbb{R}^2$ and $\mathbb{R}^3$), starting from the optimal property of Hermite curves obtained in Zhang et al. (2001).

For Hermite-type cubic splines $s \in C^2[a, b]$, the smoothness conditions lead to the $n - 1$ equations:

$$\frac{1}{h_i} \cdot m_{i-1} + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) \cdot m_i + \frac{1}{h_{i+1}} \cdot m_{i+1} = \frac{3(y_i - y_{i-1})}{h_i^2} + \frac{3(y_{i+1} - y_i)}{h_{i+1}^2}, \quad i = \overline{1, n-1} \tag{5}$$

and for a complete determination of the values m_i, $i = \overline{0, n}$, two additional end conditions are needed. These can be: the well-known Holladay's natural end conditions $s''(a) = s''(b) = 0$ (see Ahlberg et al., 1967 and Micula and Micula, 1999); clamped-end conditions $s'(a) = f'(a)$, $s'(b) = f'(b)$, when the values $f'(a)$ and $f'(b)$ are known (see Kershaw, 1972); the classical De Boor's not-a-knot end conditions (see Micula and Micula, 1999), or the end conditions proposed by Behforooz and Papamichael, (1979b, 1980) that generates the $E(\alpha)$ cubic splines (see Behforooz and Papamichael, 1979a, 1979b, 1980; Behforooz, 1995, and Papamichael and Worsey, 1981). It is proved in Behforooz (1995), that $E(2)$ cubic spline is exactly the not-a-knot cubic spline and the best $E(\alpha)$ cubic spline is $E(3)$ having superconvergence properties (see Behforooz and Papamichael, 1979a, 1980, and Papamichael and Worsey, 1981).

In this paper, preserving the values m_i, $i = \overline{2, n-2}$, be given by the Akima's procedure (3), we define the notion of partial quadratic oscillation in average and uniquely determine the left four values m_0, m_1, m_{n-1}, m_n such that the partial quadratic oscillation in average to be minimized on the end subintervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$. Since the degree of freedom