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Smooth surfaces from rational bilinear patches ☆

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A B S T R A C T

Smooth freeform skins from simple panels constitute a challenging topic arising in contemporary architecture. We contribute to this problem area by showing how to approximate a negatively curved surface by smoothly joined rational bilinear patches. The approximation problem is solved with help of a new computational approach to the hyperbolic nets of Huhnen-Venedey and Rörig and optimization algorithms based on it. We also discuss its limits which lie in the topology of the input surface. Finally, freeform deformations based on Darboux transformations are used to generate smooth surfaces from smoothly joined Darboux cyclide patches; in this way we eliminate the restriction to surfaces with negative Gaussian curvature.

1. Introduction

The growing interest of contemporary architects in freeform structures poses new challenges to digital design. In this area, there are numerous open problems, one of them being the production of smooth architectural freeform skins (surfaces). To make them affordable, they have to be composed of simple and easily manufacturable panels (patches). The manufacturing cost of a panel depends on the material and the process to produce it, which includes the production of the mold.

Let us look at an example: To produce a curved glass-fibre reinforced concrete panel, a mold is produced from styrofoam. The cheapest way of mold production is to use heated wire cutting, which can only produce ruled surfaces. Hence ruled panels are an advantage. Ruled surfaces are also advantageous for the manufacturing of form work on which the concrete is poured, for substructures and for timber constructions.

Hence, as a contribution towards *fabrication-aware design*, we ask ourselves how to compose smooth surfaces from simple ruled patches. The simplest case, smooth surfaces from bilinear patches, has been investigated recently by Käferböck and Pottmann (2013). The resulting surfaces possess rather strong shape restrictions as they represent discrete versions of affine minimal surfaces previously introduced by Craizer et al. (2010).

Can we do more with *rational* bilinear patches? It is clear that smoothly joined negatively curved patches will only generate models of negatively curved surfaces S. It turns out that the only remaining restriction is on the topology of S: the surface S has to be simply connected. As in many other instances of geometric problems in freeform architecture (Wallner and Pottmann, 2011), we are again led to discrete differential geometry: Useful discrete representations of negatively curved

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surfaces are discrete asymptotic parameterizations, namely quad meshes with planar vertex stars (*A-nets*; see e.g. Bobenko and Suris, 2008). Huhnen-Venedey and Rörig (2013) could show that simply connected A-nets (whose extraordinary vertices are of even valence) can be extended to smooth surfaces via rational bilinear patches (under a certain mild condition on the way how the quad strips in the mesh are twisted, which for a sufficiently fine A-net is satisfied anyway). They call the resulting structures *hyperbolic nets* (*H-nets*), as they are typically composed of hyperboloid patches (but may contain patches of hyperbolic paraboloids as well).

In the present paper, we further elaborate on hyperbolic nets by providing a new elementary derivation: While Huhnen-Venedey and Rörig (2013) work in the Plücker quadric model of line geometry, we just use the rational Bézier form and thus also contribute to a simple computation based on standard CAGD methods. Moreover, we add a proper algorithmic treatment of surface approximation with H-nets, formulated within a global geometric optimization framework.

1.1. Previous work

Paneling freeform skins is an important issue in architecture. Eigensatz et al. (2010) presented an optimization framework which allows one to achieve a good balance between production cost and skin quality, the latter being evaluated through the size of gaps and the kink angles occurring at adjacent panels. Ideally, these measures should be zero, but this is in general not achievable, unless the geometry and the panel seam layout are special.

Negatively curved smooth surfaces which are composed of ruled surface strips, have been addressed by Flöry (2010), Flöry and Pottmann (2010), Flöry et al. (2012). This work does not employ rational bilinear patches. It can be seen as a computational approach to semidiscrete asymptotic parameterizations, which have been investigated from a mathematical viewpoint by Wallner (2012).

As we deal here with a smooth extension of a mesh, we point to the related problem of extending the vertices of a circular mesh (quad mesh all whose quads possess a circum-circle) smoothly with Dupin cyclide patches (Bobenko and Huhnen-Venedey, 2012); a computational treatment of these cyclidic nets along with generalizations aiming at substructures in freeform architecture, has been presented by Bo et al. (2011). The theoretical beauty of the approach by Huhnen-Venedey and Rörig (2013) lies in the use of the Plücker quadric model, which reveals the close relation to cyclidic nets, in accordance with the famous line-sphere transformation of S. Lie (which is of theoretical interest, but not directly applicable as it is not real). Independently from our work, Huhnen-Venedey and Schief (2013) developed an affine approach to H-nets which is related to ours and which they use for the study of Weingarten transformations.

While cyclidic nets are always arranged along curvature lines, we may ask for generalizations by replacing Dupin cyclides by Darboux cyclides (Pottmann et al., 2012). Like Dupin cyclides, the latter also contain families of circles (up to six!), which is an advantage for production on the large architectural scale. As a first step towards the most general Darboux cyclide patchworks we address here a simple computation from hyperbolic nets via so-called Darboux transformations.

1.2. Contributions and overview

The contributions of the present paper are as follows:

1. We provide a simple CAGD proof of the main result of Huhnen-Venedey and Rörig (2013) along with formulae for computations based on the rational Bézier form (Section 2).
2. We show how to approximate a given negatively curved surface with a smooth union of rational bilinear patches, through a numerical optimization algorithm which combines A-net generation and its smooth extension with ruled quadric patches (Section 3).
3. We apply Darboux transformations to obtain smooth surfaces from Darboux cyclide patches; these surfaces are no longer restricted to negative Gaussian curvature (Section 4).

2. An elementary approach to hyperbolic nets

2.1. Some basics on rational bilinear patches

A rational bilinear patch with control points $\mathbf{b}_0, \ldots, \mathbf{b}_3$ and positive weights $w_0, \ldots, w_3$ is given by the parametric representation

$$\mathbf{x}(u, v) = \frac{(1-u)(1-v)w_0\mathbf{b}_0 + u(1-v)w_1\mathbf{b}_1 + (1-u)v w_3\mathbf{b}_3 + uvw_2\mathbf{b}_2}{(1-u)(1-v)w_0 + u(1-v)w_1 + (1-u)v w_3 + uvw_2}, \tag{1}$$

with $(u, v) \in [0, 1]^2$. One can obtain it from a bilinear patch $\bar{\mathbf{x}}(u, v)$ in $\mathbb{R}^4$, with vertices $\bar{\mathbf{b}}_i := (w_i\mathbf{b}_i, w_i)$, by mapping it into 3-space with help of the canonical projection $(x_1, \ldots, x_4) \mapsto (x_1/x_4, x_2/x_4, x_3/x_4)$. The patch lies on a ruled quadric and carries two families of straight line segments (rulings), namely the u- and v-isoparameter lines.

Let us first discuss why it is *sufficient to set three of the four weights equal to* 1, say

$$w_0 = w_1 = w_3 = 1, \tag{2}$$