Computer Aided Geometric Design 31 (2014) 27–42

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Rotation-minimizing osculating frames ☆

Rida T. Farouki [a,*], Carlotta Giannelli [b], Maria Lucia Sampoli [c], Alessandra Sestini [d]

[a] Department of Mechanical and Aerospace Engineering, University of California, Davis, CA 95616, USA
[b] Institut für Angewandte Geometrie, Johannes Kepler Universität Linz, Altenberger Strasse 69, A-4040 Linz, Austria
[c] Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Pian dei Mantellini 44, 53100 Siena, Italy
[d] Dipartimento di Matematica e Informatica, Università degli Studi di Firenze, Viale Morgagni 67a, 50134 Firenze, Italy

ARTICLE INFO

Article history:
Received 1 February 2013
Received in revised form 5 September 2013
Accepted 26 November 2013
Available online 4 December 2013

Keywords:
Rotation-minimizing frames
Pythagorean-hodograph curves
Quaternions
Angular velocity
Rigid body motion
Ruled surfaces

ABSTRACT

An orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is *rotation-minimizing* with respect to $\mathbf{f}_i$ if its angular velocity $\boldsymbol{\omega}$ satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_i \equiv 0$ — or, equivalently, the derivatives of $\mathbf{f}_j$ and $\mathbf{f}_k$ are both parallel to $\mathbf{f}_i$. The Frenet frame $(\mathbf{t}, \mathbf{p}, \mathbf{b})$ along a space curve is rotation-minimizing with respect to the principal normal $\mathbf{p}$, and in recent years *adapted* frames that are rotation-minimizing with respect to the tangent $\mathbf{t}$ have attracted much interest. This study is concerned with rotation-minimizing *osculating* frames $(\mathbf{f}, \mathbf{g}, \mathbf{b})$ incorporating the binormal $\mathbf{b}$, and osculating-plane vectors $\mathbf{f}$, $\mathbf{g}$ that have no rotation about $\mathbf{b}$. These frame vectors may be defined through a rotation of $\mathbf{t}$, $\mathbf{p}$ by an angle equal to minus the integral of curvature with respect to arc length. In aeronautical terms, the rotation-minimizing osculating frame (RMOF) specifies *yaw-free* rigid-body motion along a curved path. For polynomial space curves possessing rational Frenet frames, the existence of *rational* RMOFs is investigated, and it is found that they must be of degree 7 at least. The RMOF is also employed to construct a novel type of ruled surface, with the property that its tangent planes coincide with the osculating planes of a given space curve, and its rulings exhibit the least possible rate of rotation consistent with this constraint.

1. Introduction

A general spatial motion of a rigid body is specified by describing its position and orientation as functions of time. A particular point of the body (e.g., the center of mass) is usually chosen to describe position. To describe orientation, the variation of an orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ embedded within the body may be specified. In general, position and orientation vary independently, but in certain motion problems they may be correlated. This study is concerned with *constrained spatial motions*, in which the instantaneous angular velocity of a rigid body is related to the geometry of its center-of-mass path.

The *Frenet frame* is the most familiar orthonormal frame on a space curve, comprising the *tangent* $\mathbf{t}$, *principal normal* $\mathbf{p}$, and *binormal* $\mathbf{b} = \mathbf{t} \times \mathbf{p}$. When the Frenet frame is used to orient a body along a path, its angular velocity $\boldsymbol{\omega}$ satisfies $\boldsymbol{\omega} \cdot \mathbf{p} \equiv 0$ — i.e., it has no component in the principal normal direction. This means that the body exhibits no instantaneous rotation about the principal normal vector $\mathbf{p}$ from point to point along the path.

In aerodynamics, an embedded frame is used (Cook, 1997) to characterize variations in the attitude of an aircraft, in terms of *roll*, *pitch*, and *yaw* axes through its center of mass — the roll (or longitudinal) axis is aligned with the fuselage;

☆ This paper has been recommended for acceptance by Johannes Wallner.

* Corresponding author.
E-mail addresses: farouki@ucdavis.edu (R.T. Farouki), carlotta.giannelli@jku.at (C. Giannelli), marialucia.sampoli@unisi.it (M.L. Sampoli), alessandra.sestini@unifi.it (A. Sestini).

0167-8396/$ – see front matter
http://dx.doi.org/10.1016/j.cagd.2013.11.003

the pitch (or lateral) axis is orthogonal to it, within the plane of the fuselage and wings; and the yaw (or vertical) axis is orthogonal to that plane. Hence, pitch and yaw correspond to up/down and left/right motions of the aircraft nose, while roll corresponds to a rotation about the fuselage.

A rigid body that maintains alignment with the Frenet frame on a given spatial path exhibits a *pitch-free* motion — it has no instantaneous rotation about the principal normal $\mathbf{p}$. For a given (smooth) path, it is also possible to construct *roll-free* and *yaw-free* rigid-body motions, characterized by no instantaneous rotation about the tangent $\mathbf{t}$ and binormal $\mathbf{b}$, respectively.

Roll-free motion, with angular velocity satisfying $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$, has recently enjoyed considerable attention. In this case, the body has no instantaneous rotation about the tangent $\mathbf{t}$ from point to point along the path. Bishop (1975) first studied *adapted* orthonormal frames $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ comprising the tangent $\mathbf{t}$ and normal-plane vectors $\mathbf{u}$, $\mathbf{v}$ that have no instantaneous rotation about $\mathbf{t}$, in lieu of $\mathbf{p}$, $\mathbf{b}$. Klok (1986) described the vectors $\mathbf{u}$, $\mathbf{v}$ as solutions to differential equations, and Guggenheimer (1989) subsequently showed that these solutions amount to defining $\mathbf{u}$, $\mathbf{v}$ by a normal-plane rotation of $\mathbf{p}$, $\mathbf{b}$ through an angle equal to minus the integral of the torsion with respect to arc length.

For polynomial or rational curves, this *rotation-minimizing adapted frame* (RMAF) is not, in general, a rational locus, and this fact has prompted many approximation schemes (Farouki and Han, 2003; Jüttler and Mäurer, 1999; Wang et al., 2008). More recently, interest has emerged in identifying curves with *rational* rotation-minimizing frames (RRMF curves), which must be *Pythagorean-hodograph* (PH) *curves* (Farouki, 2008), since only PH curves possess rational unit tangents. The *Euler–Rodrigues frame* (ERF), a rational adapted frame defined on any PH curve, is a useful intermediary in identifying RRMF curves (Choi and Han, 2002; Han, 2008). The simplest non-planar RRMF curves form a subset of the PH quintics (Farouki et al., 2009a), characterized by the satisfaction of simple constraints on the curve coefficients (Farouki, 2010). Further details on the basic theory, properties, and applications of RRMF curves can be found in Barton et al. (2010), Farouki et al. (2012), Farouki and Sakkalis (2010, 2012).

Among the spatial motions mentioned above, in which angular velocity is correlated with local path geometry, the least-studied is the case of yaw-free motion satisfying $\boldsymbol{\omega} \cdot \mathbf{b} \equiv 0$. In yaw-free motion, the orientation of the body is specified by a frame $(\mathbf{f}, \mathbf{g}, \mathbf{b})$ that retains the binormal $\mathbf{b}$, but the tangent and principal normal $\mathbf{t}$, $\mathbf{p}$ are replaced by osculating-plane vectors $\mathbf{f}$, $\mathbf{g}$ that have no instantaneous rotation about $\mathbf{b}$. It is shown below that the vectors $\mathbf{f}$, $\mathbf{g}$ can be defined through an osculating-plane rotation of $\mathbf{t}$, $\mathbf{p}$ by an angle equal to minus the integral of the curvature with respect to arc length. We call the frame $(\mathbf{f}, \mathbf{g}, \mathbf{b})$ a *rotation-minimizing osculating frame* (RMOF).

The plan for the remainder of this paper is as follows. Section 2 describes the concept of rotation-minimizing frames on curves, and briefly mentions how it can also be used to identify certain special curves on a smooth surface (geodesics, lines of curvature, and asymptotic lines). Section 3 addresses the problem of *rational* rotation-minimizing frames — after briefly reviewing rational RMAFs, a detailed analysis of rational RMOFs on cubic and quintic space curves with rational Frenet frames is presented. Section 4 employs the RMOF to construct ruled surfaces interpolating a space curve, with tangent planes matching the osculating planes of that curve, and derives some useful properties of such surfaces. Finally, Section 5 summarizes the key results of this study, and identifies some open problems.

2. Rotation-minimizing frames on curves

The *parametric speed* of a differentiable space curve $\mathbf{r}(\xi)$ is defined by

$$\sigma(\xi) = \left|\mathbf{r}'(\xi)\right| = \mathrm{d}s/\mathrm{d}\xi,$$

where s is the cumulative arc length of $\mathbf{r}(\xi)$, measured from some fixed point. The curve $\mathbf{r}(\xi)$ is *regular* if its parametric speed satisfies $\sigma(\xi) \neq 0$ for all ξ. The variation of an orthonormal frame $(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi))$ defined along $\mathbf{r}(\xi)$ may be specified by its angular velocity $\boldsymbol{\omega}(\xi)$ through the relations

$$\mathbf{e}_1' = \sigma\boldsymbol{\omega} \times \mathbf{e}_1, \qquad \mathbf{e}_2' = \sigma\boldsymbol{\omega} \times \mathbf{e}_2, \qquad \mathbf{e}_3' = \sigma\boldsymbol{\omega} \times \mathbf{e}_3. \tag{1}$$

Since $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ comprise a basis for $\mathbb{R}^3$ we can write

$$\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3, \tag{2}$$

and hence the relations (1) become

$$\mathbf{e}_1' = \sigma(\omega_3\mathbf{e}_2 - \omega_2\mathbf{e}_3), \qquad \mathbf{e}_2' = \sigma(\omega_1\mathbf{e}_3 - \omega_3\mathbf{e}_1), \qquad \mathbf{e}_3' = \sigma(\omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2). \tag{3}$$

For a given *reference direction*, specified by a unit vector field $\mathbf{c}(\xi)$ along $\mathbf{r}(\xi)$, one may characterize frames $(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi))$ that are *rotation-minimizing* with respect to $\mathbf{c}(\xi)$ as follows.

Definition 1. The frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is *rotation-minimizing* with respect to $\mathbf{c}$ if its angular velocity $\boldsymbol{\omega}$ has no component in the direction of $\mathbf{c}$, i.e., $\boldsymbol{\omega} \cdot \mathbf{c} \equiv 0$.

Now if the frame vector $\mathbf{e}_1(\xi)$ is chosen as the reference direction, the rotation-minimizing frame (RMF) satisfies $\omega_1 \equiv 0$ in (2). Eqs. (3) then yield an alternative characterization, in terms of the derivatives of $\mathbf{e}_2(\xi)$ and $\mathbf{e}_3(\xi)$.