Computer Aided Geometric Design 31 (2014) 43–56

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Dual representation of spatial rational Pythagorean-hodograph curves ☆

Jernej Kozak [a,b], Marjeta Krajnc [a,b,*], Vito Vitrih [c,d]

[a] FMF, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia
[b] IMFM, Jadranska 19, Ljubljana, Slovenia
[c] IAM, University of Primorska, Muzejski trg 2, Koper, Slovenia
[d] FAMNIT, University of Primorska, Glagoljaška 8, Koper, Slovenia

ARTICLE INFO

Article history:
Received 5 February 2013
Received in revised form 5 December 2013
Accepted 7 December 2013
Available online 17 December 2013

Keywords:
Pythagorean-hodograph
Rational curves
Dual parametric curve form
Frenet frame
Quaternions

ABSTRACT

In this paper, the dual representation of spatial parametric curves and its properties are studied. In particular, rational curves have a polynomial dual representation, which turns out to be both theoretically and computationally appropriate to tackle the main goal of the paper: spatial rational Pythagorean-hodograph curves (PH curves). The dual representation of a rational PH curve is generated here by a quaternion polynomial which defines the Euler–Rodrigues frame of a curve. Conditions which imply low degree dual form representation are considered in detail. In particular, a linear quaternion polynomial leads to cubic or reparameterized cubic polynomial PH curves. A quadratic quaternion polynomial generates a wider class of rational PH curves, and perhaps the most useful is the ten-parameter family of cubic rational PH curves, determined here in the closed form.

1. Introduction

Polynomial Pythagorean-hodograph curves are characterized by the property that the Euclidean norm of their hodograph is a polynomial, not a square root of a polynomial. These curves thus have a rational unit vector field of tangents, rational offset curves, and a polynomial arc length what makes them an important practical tool that finds its applications in robotics, in CAD/CAM systems, in animations, etc. Polynomial PH curves were introduced in Farouki and Sakkalis (1990) and have widely been studied since then (see Farouki, 2008 and the references therein). The usual approach to obtain a polynomial PH curve is to integrate the appropriate hodograph constructed with the help of the complex or the quaternion polynomials in the plane or in the space case respectively (see e.g. Farouki, 2008, 1994; Choi et al., 2002).

But the natural extension of the PH property to rational curves turned out to be quite a task, and only few results are known in this direction. The main obstacle is the fact that the polynomial preimage approach cannot be applied here since the integral of a rational curve is not a rational curve in general. Planar rational PH curves were derived in Pottmann (1995a), and independently in Fiorot and Gensane (1994). The suggested construction determines a planar rational curve as the envelope of a one-parameter family of tangent lines, given by a rational unit vector field of tangents, and a rational support function which defines the distance of the tangent line from the origin. As observed in Pottmann (1995a), the introduced dual form of a planar rational PH curve turns out to be more appropriate from the computational point of view than the corresponding point representation. Interpolation schemes involving planar PH curves can be found in Pottmann (1995b).

☆ This paper has been recommended for acceptance by Rida Farouki.
* Corresponding author at: FMF, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia.
E-mail address: marjetka.krajnc@fmf.uni-lj.si (M. Krajnc).

0167-8396/$ – see front matter
http://dx.doi.org/10.1016/j.cagd.2013.12.001

A short note (Pottmann, 1994) made the first step to spatial rational PH curves. It introduced the space curves in dual representation as well as its wedge product notation. Just recently, a comprehensive study of these curves has been carried out in Farouki and Šír (2011), where the construction of rational spatial PH curves has been presented and further justified and illuminated from several equivalent geometric viewpoints. Basically, the approach originates from the implicit curve representation involving the curve binormal direction and a rational function that determines the signed distance of the osculating plane from the origin. In order to assure the PH property of the curve, its binormal directions are generated from a rational vector field of unit length tangents.

In this paper a further insight into rational spatial PH curves is provided, with emphasis to their construction in a form that could be used in practical applications. First of all, since the construction in Farouki and Šír (2011) leads to rational PH curves of high degree in general, we stick to the dual form introduced in Pottmann (1994, 1995a). Note that this curve representation can be naturally extended to any dimension. In the PH case, the dual form enables one to deal in general with polynomials of a significantly lower degree than in the curve closed form point representation. The exception is the cubic case where degrees of both representations are equal. Equipped with the dual approach, we obtain the dual PH curve from the Euler–Rodrigues frame in a similar way as already in Farouki and Šír (2011). The E–R frame is generated by quaternion polynomials, and we focus on the question, how one should choose superfluous parameters to assure that the corresponding dual form would be of a low degree. Quaternion polynomials of degree $\leqslant 2$ are considered here in detail. It is shown that linear quaternion polynomials give rise to cubic or reparameterized cubic polynomial PH curves. Based on quadratic quaternion polynomials we derive rational PH curves with the dual form of degree $m = 3, 4, 5, 6$ having $2m + 4$ degrees of freedom. In particular, a ten parametric family of cubic rational PH curves with nonconstant denominator is presented in a closed form. This comes somewhat as a surprise since there are no additional free parameters in comparison to the cubic polynomial case.

The paper is organized as follows. In the next section the dual form of a parametric space curve is introduced and its properties with the emphasis on rational curves analysed. Section 3 introduces rational PH curves with the dual form based upon E-R frame. Degree of the dual form of a rational PH curve and its reduction is considered in Section 4. In the next section curves that arise from a linear quaternion polynomial are treated. Rational PH curves from a quadratic quaternion polynomial are considered in Section 6 together with a few examples. In the end, we discuss possible future work directions.

2. Spatial curves in dual form

In Pottmann (1994) or Farouki and Šír (2011), the key step to the construction of rational PH curves was a nice implicit representation of a parametric curve. Since the approach works in the planar case too, and it could be extended to more than three dimensions if needed, we briefly recall it. Let $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ be a smooth parametric curve such that the derivatives $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are linearly independent on the parameter interval $[\alpha, \beta]$. Then the corresponding Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is well defined for each $t \in [\alpha, \beta]$. Here, the vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ denote the unit tangent, the principal normal and the unit binormal respectively. Further, a point $\boldsymbol{r}(t)$ can be uniquely recovered as the intersection of the osculating, the rectifying, and the normal plane at a particular parameter value $t \in [\alpha, \beta]$. This gives $\boldsymbol{r}$ as a set of points $\boldsymbol{p} \in \mathbb{R}^3$ that satisfy the linear system

$$\boldsymbol{b}(t) \cdot \big(\boldsymbol{p} - \boldsymbol{r}(t)\big) = 0, \qquad \boldsymbol{n}(t) \cdot \big(\boldsymbol{p} - \boldsymbol{r}(t)\big) = 0, \qquad \boldsymbol{t}(t) \cdot \big(\boldsymbol{p} - \boldsymbol{r}(t)\big) = 0, \quad t \in [\alpha, \beta]. \tag{1}$$

Here, the dot $\cdot$ denotes the scalar product in $\mathbb{R}^3$. If the torsion τ of the curve $\boldsymbol{r}$ does not vanish on $[\alpha, \beta]$, as observed in Farouki and Šír (2011), the one-parametric family of linear systems (1) is by Frenet–Serret formulas equivalent to

$$\boldsymbol{b}^{(\ell)}(t) \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})^{(\ell)}(t) = 0, \quad \ell = 0, 1, 2,\ t \in [\alpha, \beta]. \tag{2}$$

These systems can further be simplified by any nonzero function $\phi \in \boldsymbol{C}^2([\alpha, \beta])$ to

$$\boldsymbol{u}^{(\ell)}(t) \cdot \boldsymbol{p} - f^{(\ell)}(t) = 0, \quad \ell = 0, 1, 2,\ t \in [\alpha, \beta], \tag{3}$$

where

$$\boldsymbol{u} := \phi \boldsymbol{b}, \qquad f := \phi \boldsymbol{b} \cdot \boldsymbol{r}. \tag{4}$$

Namely, (2) and (3) are equivalent since by the Leibniz rule one has

$$\begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{p} - f \\ \boldsymbol{u}' \cdot \boldsymbol{p} - f' \\ \boldsymbol{u}'' \cdot \boldsymbol{p} - f'' \end{pmatrix} = \begin{pmatrix} \phi & 0 & 0 \\ \phi' & \phi & 0 \\ \phi'' & 2\phi' & \phi \end{pmatrix} \begin{pmatrix} \boldsymbol{b} \cdot \boldsymbol{p} - \boldsymbol{b} \cdot \boldsymbol{r} \\ \boldsymbol{b}' \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})' \\ \boldsymbol{b}'' \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})'' \end{pmatrix}.$$

Note that $\frac{f}{\phi}$ denotes the signed distance of the osculating plane with the normal vector $\boldsymbol{u}$ from the origin. If $\boldsymbol{u}$ and f are given, and $\det(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}'') \neq 0$, the curve $\boldsymbol{r}$ may be determined from (3). Rather then using the closed form solution (Farouki and Šír, 2011) that gives the point representation

$$\boldsymbol{r} = \frac{1}{\det(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}'')} \big(f\, \boldsymbol{u}' \times \boldsymbol{u}'' + f'\, \boldsymbol{u}'' \times \boldsymbol{u} + f''\, \boldsymbol{u} \times \boldsymbol{u}'\big),$$