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A new four-point shape-preserving C^3 subdivision scheme ☆

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ABSTRACT

A new binary four-point subdivision scheme is presented, which keeps the second-order divided difference at the old vertices unchanged when the new vertices are inserted. Using the symbol of the subdivision scheme, we show that the limit curve is at least C^3 continuous. Furthermore, the conditions imposed on the initial points are also discussed, thus the given limit functions are both monotonicity preserving and convexity preserving

1. Introduction

Subdivision methods are widely used in many areas including CAGD, CG and related areas. Most of the existing univariate subdivision schemes are binary, stationary and linear. The classical binary four-point scheme (Dubuc, 1986; Dyn et al., 1987) is one of the earliest and most popular interpolatory curve subdivision schemes, which can only achieve C^1 continuous curves. In Dyn et al. (1999) the convexity is in the functional sense and is achieved for data satisfying certain conditions in addition to the convexity conditions. Many subdivision schemes cannot preserve monotonicity in the current literature. Cai (1995) presented a four-point interpolatory subdivision scheme which is C^1 continuous in nonuniform control points and discussed the monotonicity preservation of the limit curve. A family of local nonlinear stationary interpolatory subdivision schemes which preserve monotonicity were constructed in Kuijt and van Damme (1999). In order to improve the smoothness of subdivision curves, Hassan et al. (2002) introduced an interpolating four-point ternary subdivision scheme, which generated C^2 continuous curves. Cai (2009) discussed the convexity preserving properties of the subdivision scheme (Hassan et al., 2002). Kuijt and van Damme (2002) were concerned with a class of shape preserving four-point subdivision scheme which is stationary and interpolated nonuniform data. However, the curves generated by the above methods are at most C^2 continuous. Hao et al. (2011) described a linear six-point binary approximating subdivision scheme which preserves convexity while its support is large. Hormann and Sabin (2008) constructed the subdivision schemes according to the properties, such as the support, the Hölder regularity, the precision set and so on. Different subdivision schemes were designed for meeting different requirements in Sabin and Dodgson (2005), Hernández-Mederos et al. (2009), Augsdörfer et al. (2010), Floater (2011). Deng and Wang (2010) took a geometric approach to generate subdivision curve, and the resulting curve is G^1 continuous and the shape is determined by the initial points and their tangent vectors. The circular arc segment was reproduced when the initial points and their initial tangent vectors were sampled from a circular arc segment.

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The important issues in the implementation of the subdivision scheme are smoothness, size of support and shape preserving properties. The goal of this paper is to construct a C^3 continuous subdivision scheme which preserves shape. Inspired by Hormann and Sabin (2008), Sabin and Dodgson (2005), Augsdörfer et al. (2010), we introduce a binary four-point C^3 continuous subdivision scheme in Section 3. In Section 4 we discuss the conditions of the initial points guaranteeing monotonicity preservation. The conditions of preserving convexity are given in Section 5. We conclude our work with a short summary in Section 6.

2. Preliminaries

Given a set of initial control points $P^0 = \{p_i^0\}_{i\in\mathbb{Z}}$, let $P^k = \{p_i^k\}_{i\in\mathbb{Z}}$ be the set of control points at order k ($k \geqslant 0$, $k \in \mathbb{Z}$) subdivision. Define $\{p_i^{k+1}\}_{i\in\mathbb{Z}}$ recursively by the following binary subdivision rules:

$$p_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} p_j^k, \quad i \in \mathbb{Z}, \tag{1}$$

where the finite set $a = \{a_i\}_{i\in\mathbb{Z}}$ is called a mask. The symbol of the scheme is defined as $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$.

Theorem 1. *(See Dyn and Levin, 2002.) Let a binary subdivision scheme S be convergent. Then the mask $a = \{a_i\}_{i\in\mathbb{Z}}$ satisfies*

$$\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1. \tag{2}$$

Theorem 2. *(See Dyn and Levin, 2002.) Let a subdivision scheme S with a mask $a = \{a_i\}_{i\in\mathbb{Z}}$ satisfy (2). Then there exists a subdivision scheme S_1 (first order difference scheme of S) which satisfies the property*

$$\mathrm{d}\mathrm{P}^k = S_1\, \mathrm{d}\mathrm{P}^{k-1},$$

where $\mathrm{P}^k = S^k \mathrm{P}^0$, $\mathrm{dP}^k = \{(\mathrm{d}p^k)_i = 2^k(p_{i+1}^k - p_i^k) \mid i \in \mathbb{Z}\}$. The symbol of S_1 is $a^{(1)}(z) = \frac{2z}{1+z} a(z)$. Generally, if S_n exists and S_n is the nth order difference scheme of S with a mask $a^{(n)} = \{a_i^{(n)}\}_{i\in\mathbb{Z}}$, then the symbol of S_n is $a^{(n)}(z) = \sum_{i\in\mathbb{Z}} a_i^{(n)} z^i = (\frac{2z}{1+z})^n a(z)$.

Theorem 3. *(See Zheng et al., 2004.) Let a subdivision scheme S have a mask $a^{(0)} = \{a_i^{(0)}\}_{i\in\mathbb{Z}}$, and its jth order difference scheme S_j ($j = 1, 2, \ldots, n+1$) exist and have the mask $a^{(j)} = \{a_i^{(j)}\}_{i\in\mathbb{Z}}$ satisfying*

$$\sum_{i\in\mathbb{Z}} a_{2i}^{(j)} = \sum_{i\in\mathbb{Z}} a_{2i+1}^{(j)} = 1, \quad j = 0, 1, 2, \ldots, n. \tag{3}$$

If there exists an integer $L \geqslant 1$, such that $\|(\frac{1}{2} S_{n+1})^L\|_\infty < 1$, then the subdivision scheme S is C^n continuous, where

$$\left\| \left(\frac{1}{2} S_{n+1}\right)^L \right\|_\infty = \left\{ \max \sum_{i\in\mathbb{Z}} \left| b_{i-2^L(n+1)}^{[L]} \right| : 0 \leqslant i < 2^L \right\},$$

$$b^{[L]}(z) = b(z) b\left(z^2\right) \cdots b\left(z^{2^{L-1}}\right), \qquad b(z) = \frac{1}{2} a^{(n+1)}(z).$$

Especially, when $L = 1$,

$$\left\| \frac{1}{2} S_{n+1} \right\|_\infty = \frac{1}{2} \max\left\{ \sum_{i\in\mathbb{Z}} \left| a_{2i}^{(n+1)} \right|, \sum_{i\in\mathbb{Z}} \left| a_{2i+1}^{(n+1)} \right| \right\}.$$

3. A four-point shape-preserving C^3 subdivision scheme

3.1. Construction of the scheme

Sabin and Dodgson (2005) observed that the second-order divided difference at each new inserted point of the four-point scheme (Dubuc, 1986) was the mean of the second-order divided differences of the adjacent old points. The limit curve generated by the subdivision scheme (Dubuc, 1986) is only C^1 continuous. A relaxation of the four-point scheme was constructed in order to make the new second-order divided difference closer to that of the old points in Hormann and Sabin (2008). The subdivision scheme (Hormann and Sabin, 2008) is C^3 continuous, while the second-order divided difference at the old points changes at different subdivision levels. Augsdörfer et al. (2010) moved the old point C such that the second-order divided difference of the modified point $d_{\hat{C}}$ is equal to a weighted mean of the old second-order divided difference d_{C^k} and the new second-order divided difference $d_{C^{k+1}}$, i.e. $d_{\hat{C}} = \alpha d_{C^k} + (1-\alpha) d_{C^{k+1}}$, where k denotes the subdivision level,