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Splines and unsorted knot sequences ☆

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Dedicated to Carl de Boor on his 75th birthday

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ABSTRACT

The definition of a B-spline is extended to unordered knot sequences. The added flexibility implies that the resulting piecewise polynomials, named U-splines, can be negative and locally linearly dependent. It is therefore remarkable that linear combinations of U-splines retain many properties of splines in B-spline form including smoothness, polynomial reproduction, and evaluation by recurrence.

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1. Introduction

A univariate spline of degree d consisting of $n+d$ polynomial pieces is defined by n B-spline coefficients and $n+d+1$ real-valued scalars t_j, called knots. By convention, these knots are listed in non-decreasing order. It is natural to ask what advantages or drawbacks arise if we drop this convention and define splines using an unsorted knot sequence. Unsorted knot sequences appear, for example, as a special case of splines with complex knots (Goldman and Tsianos, 2011) and arise in interpolation problems for linear reproduction, as shown in Example 3.2 of this paper. One approach to splines, polar forms (Ramshaw, 1989), lists ordering-independence as one of its fundamental properties; however, all detailed discussions and lecture notes assume either sorted knots or non-repeated knots. Of the many interactive online illustrations of splines only a small number, four at current counting, allows manipulating the knots. Of these, all but one either block knots from passing one another or re-sort the knot sequence (e.g. the nice applet (Kilian et al., 2013)). Only one illustration (Kraus, 2013) allows for full flexibility of knots, but encounters singularities. Apart from applications, the motivation for exploring U-splines is to probe and re-enforce the notions, proofs and properties underlying splines.

The first hint of what splines for unordered knot sequences might look like arose from a Gedankenspiel by Carl de Boor, Malcolm Sabin and the author a decade ago, that, for $t_j > t_{j+1}$, the analogue of a B-spline of degree 0 might be defined as the negative of the characteristic function of the interval $[t_{j+1} \,..\, t_j]$. Ten years on, this ansatz is substantiated by U-splines.

Overview. In Section 2, U-splines are defined for unsorted, but 'collocated' knot sequences. A collocated knot sequence repeats knot values only consecutively or sufficiently spaced apart. As is customary for B-splines, the ith U-spline of degree d is associated with the subsequence $t_{i:i+d+1} := (t_i, t_{i+1}, \ldots, t_{i+d+1})$, but now of the unsorted knot sequence $t_{0:n+d}$. A useful insight, Lemma 2.1, is that any U-spline can be expressed as a multiple of the B-spline with the same, but sorted

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subsequence of knots. Alternatively we derive, again for unsorted, collocated knot sequences, a U-spline of higher degree by recurrence from U-splines of lower degree. The recurrence is that of B-splines, but it starts from signed characteristic functions.

In Section 3 we characterize the space spanned by U-splines. To this end, we need to 'complete' finite knot sequences and clarify, for U-splines, the notion of 'basic interval'. The notion appears, in deceptively simple form, when analyzing splines in B-spline form. On a basic interval, the U-splines form a partition of unity, Marsden's identity holds, they have linear precision and we can compute derivatives via differences of coefficients. But since U-splines may take on negative values, they can be locally linearly dependent.

Section 4 shows that a slight variant of de Boor's algorithm evaluates splines with collocated, unsorted knots. Prior to evaluation, any U-spline whose first knot equals its last knot has to be removed and, in the final step of the modified algorithm, the values of multiple intervals containing the point of evaluation have to be summed. Knot insertion is well-defined but does not imply variation diminution as for B-splines. Finally, Section 5 gives an example of knots that are not collocated. For the example no two-term evaluation recurrence with convex weights can correctly evaluate the associated spline. This points to collocated sequences as a maximal practical generalization of non-decreasing knot sequences.

2. Splines with unsorted, collocated knots

We start by characterizing an important sub-class of all unsorted knot sequences.

Definition 2.1 *(Collocated knot sequence).* Let $t_{0:n+d}$ be an unsorted sequence of real scalars called *knots* hereafter. This knot sequence is *d-collocated* if

$$\forall 0 < j \leqslant d, \quad t_i = t_{i+j} \quad \Rightarrow \quad t_i = t_{i+1} = \cdots = t_{i+j}. \tag{L}$$

That is, if any knot of a collocated sequence is repeated after fewer than d entries then all intermediate knots must have the same value.

Whenever the degree d is clear from the context, we simply say that the knot sequence is collocated. A collocated sequence appears to be necessary to establish basic properties and apply algorithms that make splines useful (see (1), (9), (18), (21), Section 5 of this paper).

For collocated knot sequences, we define U-splines analogous to B-splines via a table of *divided differences* $\Delta(t_{i:j})$. For a sufficiently smooth univariate real-valued function h with kth derivative $D^k h$,

$$\Delta(t_i)h := h(t_i),$$
$$\Delta(t_{i:j})h := \begin{cases} (\Delta(t_{i+1:j})h - \Delta(t_{i:j-1})h)/(t_j - t_i), & \text{if } t_i \neq t_j, \\ \frac{D^{j-i}h}{(j-i)!}(t_i), & \text{if } t_i = t_j, \end{cases} \quad j \in \{i+1, \ldots, i+d+1\}. \tag{1}$$

Since for collocated knots, $t_i = t_j$ implies that $t_i = t_{i+1} = \cdots = t_j$, the definition is consistent in that the second case of (1) is the limiting case of the first when $t_i \to t_j$.

Definition 2.2 *(U-spline from divided differences).* Let $t_{i:i+d+1}$ be an unsorted, d-collocated knot sequence. Then the piecewise polynomial of degree d defined by

$$U(x|t_{i:i+d+1}) := (t_{i+d+1} - t_i)\, \Delta(t_{i:i+d+1})\big(\max\{(\cdot - x), 0\}\big)^d, \tag{2}$$

is called a *U-spline*.

If $t_{i:j}$ is a non-decreasing sequence then its knots are automatically collocated and $U(x|t_{i:j})$ is a B-spline $B(x|t_{i:j})$ as defined in de Boor (2001, IX(2)).

The constructive definition of divided differences is typically arranged in the form of a divided difference table as in recurrence (1). An alternative is to define $\Delta(t_{i:j})h$ as the leading coefficient of the polynomial that interpolates h at the knots $t_{i:j}$ (Conte and de Boor, 1980, p. 42). This equivalent definition is clearly invariant under re-ordering of $t_{i:j} := (t_i, \ldots, t_j)$. Therefore, together with (2), we have the following useful Lemma 2.1, first suggested in this form by Carl de Boor.

Lemma 2.1 *(Re-ordering of knots). If $s_{i:j}$ is any re-ordering of $t_{i:j}$ then*

$$(s_j - s_i)U(x|t_{i:j}) = (t_j - t_i)U(x|s_{i:j}). \tag{3}$$

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