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Technical Section

# Approximating implicit curves on plane and surface triangulations with affine arithmetic ☆

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ABSTRACT

We present a spatially and geometrically adaptive method for computing a robust polygonal approximation of an implicit curve defined on a planar region or on a triangulated surface. Our method uses affine arithmetic to identify regions where the curve lies inside a thin strip. Unlike other interval methods, even those based on affine arithmetic, our method works on both rectangular and triangular decompositions and can use any refinement scheme that the decomposition offers.



## 1. Introduction

The numerical solution of systems of non-linear equations in several variables is a key tool in geometric modeling and computer-aided geometric design [1]. In many applications, such as surface intersection and offset computation, the solution is not a set of isolated points but rather a curve or a surface. The simplest case is the solution of an equation $f(x,y)=0$, which gives an implicit curve on the plane.

Computing a polygonal approximation of an implicit curve is a challenging problem because it is difficult to find points on the curve and also because the curve may have several connected components. Therefore, robust approximation algorithms must explore the whole region of interest to avoid missing any components of the curve. One approach for achieving robustness is to use interval methods [2,3], which are able to probe the behavior of a function over whole regions instead of relying on point sampling. Interval methods lead naturally to spatially adaptive solutions that concentrate efforts near the curve.

Several interval methods have been proposed for robustly approximating an implicit curve on the plane (see Section 2). These methods explore a rectangular region of interest by decomposing it recursively and adaptively with a quadtree and using interval estimates for the values of $f$ (and sometimes of its gradient) on a cell as a subdivision criterion.

Affine arithmetic (AA) [4] is a generalization of classical interval arithmetic that explicitly represents first-order partial correlations, which can improve the convergence of interval estimates. Some methods have used AA for approximating implicit curves, successfully exhibiting improved convergence, but none has exploited the additional geometric information provided by AA and none has worked on triangulations. Indeed, while all interval methods can compute interval estimates on rectangular cells, classical interval arithmetic cannot handle triangles naturally, except by enclosing them in axis-aligned rectangles. Thus, existing interval methods are restricted to rectangular regions. Moreover, to handle implicit curves on triangulated surfaces, these methods would have to use a 3d axis-aligned box containing each triangle, which is wasteful.

In this paper, we describe an interval method for adaptively approximating an implicit curve on a refinable triangular decomposition of the region of interest. Our method uses the geometric information provided by AA as a flatness criterion to stop the recursion and is thus both spatially and geometrically adaptive in the sense of Lopes et al. [5]. Our method can handle implicit curves given by algebraic or transcendental formulas, works on triangulated plane regions and surfaces of arbitrary genus, and can use any mesh refinement scheme. Fig. 1 shows an example of our method in action on a triangulated surface. Note how the mesh is refined near the implicit curve.

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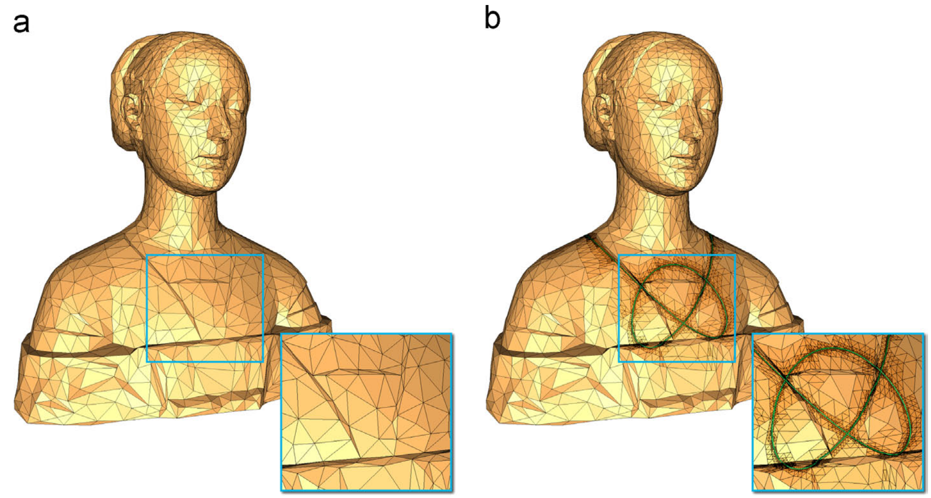


**Fig. 1.** Our method in action for the *knot* curve given implicitly by $y^2(3+2y)-(x^2-1)^2=0$ on an unstructured triangle mesh: (a) input coarse mesh and (b) adaptively refined mesh and polygonal approximation (green) computed by our method. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

After briefly reviewing some of the related work in Section 2 and the main concepts of AA in Section 3, we explain in detail in Section 4 how to use AA to extract geometric estimates in the form of strips for the location of the curve in a triangle. This is the basis of an interval method that can be used on triangulations, both on the plane and on surfaces, which we present in Section 5. We discuss some examples of our method in action in Section 6 and we report our conclusions and suggest directions for future work in Section 7.

A previous version of this paper [6] focused on plane curves only. Here, we focus on curves on surfaces. We also discuss plane curves for motivation, simplicity of exposition, and completeness. In addition to the material on surfaces presented in Sections 4.4 and 6, we include a performance comparison of the strategies for handling triangles with AA in Section 4.3 and an expanded and detailed explanation of how our method works in Section 5.

## 2. Related work

Dobkin et al. [7] described in detail a continuation method for polygonal approximation of implicit curves in regular triangular grids generated by reflections. Since the grid is regular, their approximation is not adaptive. The selection of the grid resolution is left to the user. Persiano et al. [8] presented a general scheme for adaptive triangulation refinement which they applied to the polygonal approximation of implicit curves in triangular grids. These two methods work well for fine grids but they cannot claim robustness since they rely on point sampling.

Suffern and Fackerell [9] were probably the first to apply interval methods for plotting implicit curves adaptively using quadtrees. Mitchell [10] revisited their work and helped to spread the word on interval methods for computer graphics.

Snyder [11,12] described a complete modeling system based on interval methods which included an adaptive algorithm for approximating implicit curves. His algorithm uses interval estimates for the gradient of the function defining the curve to incorporate global parametrizability in the subdivision criteria. The leaf cells in the resulting decomposition can vary in size, even though the approximation is not explicitly adapted to the curvature of the curve.

Lopes et al. [5] presented an interval method for polygonal approximation of implicit curves that uses interval estimates of the gradient for finding an approximation that is both spatially and geometrically adaptive, in the sense that it uses larger cells when the curve is approximately flat. Their method is in the same spirit as ours, except that it works only with rectangular quadtrees on the plane and relies on automatic differentiation, which can be avoided by using AA, as we shall show.

Comba and Stolfi [13] introduced AA and showed an example of how it can perform better than classical interval arithmetic when plotting implicit curves. Further examples were given by Figueiredo and Stolfi [14]. Martin et al. [15] compared the performance of several interval methods for plotting algebraic curves using quadtrees, including methods based on AA and variants. None of these papers exploited the additional geometric information provided by AA. This has been done for ray tracing implicit surfaces by Cusatis et al. [16] and for approximating parametric curves by Figueiredo et al. [17], but as far as we know has not yet been done for polygonal approximation of implicit curves, either on the plane or on surfaces.

Bühler [18,19] proposed a cell pruning method based on a linearization of implicit objects derived from AA. In addition to reducing the number of enclosure cells, her method provides a tight piecewise linear covering adapted to the topology of the object instead of a covering using overestimated axis-aligned bounding boxes. This approach is in the same spirit as our own, but it uses rectangular cells only and can generate approximations with cracks across cells.

## 3. Affine arithmetic

We now briefly review the main concepts of AA. For details, see [20] and [4].

Like classical interval arithmetic [2,3], affine arithmetic is an *extended arithmetic*: it represents real quantities with more than just one floating-point number; it provides replacements for the standard arithmetic operations and elementary functions that work on such extended representations; and it is able to extract information on the range of computed quantities from the extended representation. The computations in both interval and affine arithmetic take into account all rounding errors in floating-point arithmetic and so provide reliable results.

Interval arithmetic uses two floating-point numbers to represent intervals containing quantities. Affine arithmetic represents a quantity $q$ with an *affine form*

$$\hat{q}=q_0+q_1\varepsilon_1+q_2\varepsilon_2+\cdots+q_n\varepsilon_n$$

where $q_i$ are real numbers and $\varepsilon_i$ are *noise symbols* which vary in the interval $[-1,1]$ and represent independent sources of uncertainty. From this representation, one deduces an *interval estimate*