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Rational splines for Hermite interpolation with shape constraints ☆

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ABSTRACT

This paper is concerned in shape-preserving Hermite interpolation of a given function f at the endpoints of an interval using rational functions. After a brief presentation of the general Hermite problem, we investigate two cases. In the first one, f and f' are given and it is proved that for any monotonic set of data, it is always possible to construct a monotonic rational function of type [3/2] interpolating those data. Positive and convex interpolants can be computed by a similar method. In the second case, results are proved using rational function of type [5/4] for interpolating the data coming from f, f' and f'' with the goal of constructing positive, monotonic or convex interpolants. Error estimates are given and numerical examples illustrate the algorithms.

1. Introduction

This paper is concerned in shape-preserving Hermite interpolation of a given function f and its derivatives at the endpoints of an interval ($[0, 1]$ is chosen for the sake of simplicity) using rational functions. After a brief presentation of the general Hermite problem, we investigate two cases.

In the first one, f and f' are given and it is proved that for any monotonic set of data, it is always possible to construct a monotonic rational function of type [3/2] interpolating those data. Positive or convex interpolants can be build by similar algorithms. This first problem has been studied by many people, in particular by Gregory and coworkers in several papers, e.g. Delbourgo and Gregory (1985), Gregory and Sarfraz (1990). Even if the results are similar, our approach is slightly different as it is based on the properties of a control polygon associated with the function.

The control polygon is again used as a tool in the second case where f, f' and f'' are given. For any positive (resp. monotonic, convex) set of data, algorithms are designed to construct a positive (resp. monotonic, convex) rational function of type [5/4] interpolating any suitable data. This second problem had not been considered in its full generality.

In the two cases, the interpolant is depending on a free parameter σ and for each subcase, we give an algorithm to obtain the solution and not only a result of existence. Error estimates are set up and numerical examples illustrate the various algorithms adapted to each problem.

We only cite some papers solving this problem by using various families of methods. They can be classified as follows: polynomials with variable degrees (Costantini, 2000), Chebyshev systems (Costantini et al., 2005), rational functions (Carnicer et al., 1996; Clements, 1990; Delbourgo and Gregory, 1985; Gregory and Sarfraz, 1990), polynomial splines (De Vore, 1977; Eisenstat et al., 1985; Schumaker, 1983) and subdivision methods (Lyche and Merrien, 2006; Merrien and Sablonnière, 2003; Pelosi and Sablonnière, 2008). A different approach for C^2 (and more) interpolation with shape constrains was proposed in Costantini et al. (2001), Manni (2001) with parametric curves.

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Here is an outline of the paper. In Section 2, we study some properties of the families $\mathcal{R}[n/n-1]$ of rational functions that are used later. For this set, we have the possibility of associating a control polygon to any function $R \in \mathcal{R}[n/n-1]$ which is obtained by reproduction of affine polynomials (this idea was introduced in Piţul and Sablonnière, 2009; Sablonnière, 2009). As usual in CAGD, thanks to the total positivity of basic functions, the shape properties of the control polygon imply those of the underlying function (see e.g. Goodman, 1996). In Section 3, we give the general solutions of various Hermite interpolation problems at endpoints of the interval $[0,1]$ by rational functions in convenient spaces $\mathcal{R}[n/n-1]$.

Sections 4 and 5 are devoted to the algorithms for shape constrained problems. As the considered problems have an infinite number of solutions, we deliberately chose the parameters in order to design simple algorithms in the forms of Lyche and Merrien (2006) for the computation of satisfying solutions. Then numerical examples test the feasibility of the methods. We have only proposed a brief approach of the case $\mathcal{R}[3/2]$ (already studied in Delbourgo and Gregory, 1985) with the solution of the monotonic problem deduced from the shape of the control polygon. For the second Hermite interpolation problem in $\mathcal{R}[5/4]$, the study is subdivided into three parts corresponding to constraints of positivity, monotonicity and convexity.

Finally, in the last section, first subsection, we give some bounds on the errors $f-R$ and $f'-R'$ for functions f having a bounded fourth (resp. sixth) derivative. In addition, we explicitly compute some values of the constants involved in majorations for both families $\mathcal{R}[3/2]$ and $\mathcal{R}[5/4]$ of rational interpolants. For $f-R$, those constants can be bounded independently from the parameter introduced in the algorithms allowing shape constrains. Then, in the second subsection, we show that, for Hermite interpolation in $\mathcal{R}[5/4]$ on the interval $[0,h]$, we can obtain an error $f-R$ in $O(h^4)$ (instead of $O(h^2)$), with our choice of the parameter σ in the monotonic case with data such that $f'(a)>0$ and $f'(a+h)>0$. A similar result was already obtained in Delbourgo and Gregory (1985) for interpolants in $\mathcal{R}[3/2]$.

It is clear that the local schemes presented in this paper can be used for piecewise Hermite interpolation with shape constraints, as done e.g. in Lyche and Merrien (2006). In addition, the latter may vary in each interval, shape constraints can be accumulated and the choice of the local rational interpolant can be adapted to each specific case.

2. Basis and control polygon

For every positive integer $n \geqslant 2$, a rational function of type $\mathcal{R}[n/n-1]$ is defined by

$$R(t)=\frac{P(t)}{Q(t)}=\frac{\sum_{i=0}^{n}\bar{w}_i c_i B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}, \tag{1}$$

where for $p=n-1$ or $p=n$, the $B_i^p(t)=\binom{p}{i}t^i(1-t)^{p-i}$, $i=0,\dots,p$, are the Bernstein polynomials in $\mathbb{P}_p$. For the sake of convenience, we set $B_i^p(t)=0$ for $i<0$ and $i>p$.

The weights $\{w_j\}_{j=0,\dots,n-1}$ of the denominator are supposed to be positive and will be the *shape parameters*. The weights $\{\bar{w}_i\}_{i=0,\dots,n}$ of the numerator are depending on the w_j's and are chosen in such a way that both monomials $m_0(x)=1$ and $m_1(x)=x$ have a rational representation of the above type (1). Following Sablonnière (2009), these weights are computed as in Proposition 1 below. The c_i's are the *control coefficients* of the rational function R.

Definition 1. A function f is *reproduced* by the representation of type $\mathcal{R}[n/n-1]$ if there exists $(\xi_i)_{i=0,\dots,n}$ such that for $t\in[0,1]$, $f(t)=\frac{\sum_{i=0}^{n}\bar{w}_i f(\xi_i)B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}$.

Proposition 1. *Both monomials m_0, m_1 are reproduced by the rational functions of type $\mathcal{R}[n/n-1]$ if and only if, for $i=0,\dots,n$,*

$$\bar{w}_i=\frac{i}{n}w_{i-1}+\left(1-\frac{i}{n}\right)w_i, \tag{2}$$

$$\xi_i=\frac{i}{n}\frac{w_{i-1}}{\bar{w}_i} \tag{3}$$

where $w_{-1}=w_n=0$.

Proof. Since for any $n,i,j\in\mathbb{N}$ and $t\in[0,1]$, we have $jB_j^n(t)=ntB_{j-1}^{n-1}(t)$ and $(n-j)B_j^n(t)=n(1-t)B_j^{n-1}(t)$, it is easy to prove that when defining $\bar{w}_i$ by (2) with $w_{-1}=w_n=0$, we obtain by degree raising

$$1\times\sum_{j=0}^{n-1}w_jB_j^{n-1}(t)=\sum_{i=0}^{n}\bar{w}_iB_i^n(t),\quad t\in[0,1].$$

Conversely, the unicity of the decomposition of $1\times\sum_{j=0}^{n-1}w_jB_j^{n-1}(t)$ in the Bernstein basis $\{B_i^n(t)\}_{i=0,\dots,n}$ gives the unicity of the $\bar{w}_i$ depending on the w_j.