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Biarcs and bilens ☆

A.I. Kurnosenko

Private Research, Russian Federation

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ABSTRACT

Biarc curves are considered from the standpoints of the theory of spirals and Möbius maps. Parametrization and reference formulae, covering the whole variety of biarcs, are proposed. A region is constructed, enclosing all spirals with common circles of curvature at the endpoints. This region, named *bilens*, is bounded by two biarcs.

1. Introduction

In articles (Kurnosenko, 2010a, 2010b) the main concepts of the theory of spiral curves have been presented as the background to the solution of the two-point G^2 Hermite interpolation problem. The current note explores further the properties of spirals. Simplest spirals, namely, biarc curves, constitute the subject matter of the article.

A special role of biarcs can be seen at the very first statements of the theory, describing end conditions, allowing a spiral to exist: if boundary circles of curvature of a spiral are tangent (Fig. 1(a)), biarc is the unique possible spiral (Kurnosenko, 2009, Th. 2). As soon as a gap occurs between these two circles, an infinite set of spirals can squeeze through it.

Biarcs appear again as the bounds of the region in which all spirals with common boundary G^2 Hermite data lie. This region is called a *bilens*; an example is AJ_1BJ_2A in Fig. 1(b). The first proof of the bounding property of bilens was given in Kurnosenko (2001, Th. 3) for small spiral arcs, close to the chord. In this note this restriction is removed.

Since biarcs were proposed in Bolton (1975), an extensive literature has evolved, which traces the role of biarcs as a flexible tool for curves interpolation. Examples of biarcs, below assigned to a class of *long spirals*, first appear in Yong et al. (1999). In the recent work Meek and Walton (2008) an attempt is made to parametrize the family of biarcs with given G^1 Hermite data. Although much has been written about biarcs, the present work extends previous and presents some new results. The emphasis is made on biarcs properties, inherited from the general properties of spirals, and on Möbius maps, which convert a spiral to a spiral, a biarc to a biarc. New features are:

1. The proposed local coordinate system can be considered as canonical for these curves: the shape parameters are separated from the positional ones. This yields a set of simple and symmetric reference formulae.
2. The whole variety of biarcs is parametrized in continuous manner in the parameter range $p \in [-\infty; +\infty]$.

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E-mail address: Alexey.K@dxdy.ru.

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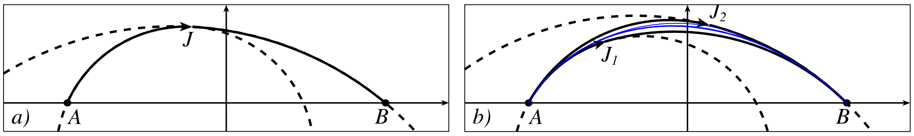

Fig. 1. Biarc AJB (made by tangent circles) and bilens (the region between two biarcs).

3. In a lot of studies only a part of biarc properties, related to the problem under study, have been described. Here we summarize the most important properties (Section 4). To obtain a description, unified for C-, S-, and J-shaped biarcs, it is sufficient *not to ignore* the natural curvature sign, and the signs of oriented angles.
4. Since strict inequalities like $0 < |\alpha| < \pi$, instead of $0 \leqslant |\alpha| \leqslant \pi$ (1), were accepted in previous studies, the families, like shown in Fig. 5(a), Fig. 6, became lost.
5. Biarc curve, being composed of two circular arcs, is a simple curve by definition. Nevertheless, it is often presented as a rather complicated object. Several cases (intervals for parameter) are enumerated in Meek and Walton (2008). In Yong et al. (1999) 128 cases are mentioned. Here two main cases $p \lessgtr 0$ differ in increasing/decreasing curvature.

Biarcs formalism is applied to define the bilens, and to prove the bilens theorem, which is an important positional inequality for spiral arcs. This result improves the bounding region, constructed in Bartoň and Elber (2011) for convex spiral arcs, to best possible (Section 5.1).

2. Notation and geometric preliminaries

By a *spiral* is meant a smooth planar curve of monotonic, piecewise continuous curvature (possibly changing sign), not containing the circumference of a circle. The main theoretical facts, resulting from the monotonicity of curvature, are quoted and illustrated in papers (Kurnosenko, 2010a, 2010b). Proofs are given in Kurnosenko (2009, 2011a, 2011b).

In this article the following notions are reused.

- *Normalized position* and *normalized curvatures* of a spiral arc (Kurnosenko, 2010a, Sec. 2.1): the chord of a normalized spiral is the segment $x \in [-1; 1]$ of the x-axis.
- *Short* (, ) and *long* (, ) spiral arcs (Kurnosenko, 2010b, Sec. 2): short spirals do not intersect the rays $\{(x, y)\colon\ |x| > 1,\ \ y = 0\}$.
- *Cumulative boundary angles*: explained in Kurnosenko (2010b, Sec. 2), illustrated below in Section 3.3. Cumulative angles keep the history of twisting of the spiral near endpoints.
- *Lens*, associated with cumulative boundary angles of the arc; explained in Kurnosenko (2010b, Sec. 2).

Main pieces of notation are listed below:

A, B, J are the startpoint $A = (-1, 0)$ and the endpoint $B = (1, 0)$ of a spiral arc. If $\overset{\frown}{AB}$ is biarc, J denotes the join point.

$\mathbf{n}(\varphi)$ is the unit vector $(\cos\varphi, \sin\varphi)$.

$\mathcal{A}(\varphi)$ denotes a circular arc, traced from the point A to B, $\mathbf{n}(\varphi)$ being the tangent at A (6). The arc $\mathcal{A}(0)$ is the chord AB itself. The arc $\mathcal{A}(\pm\pi)$, passes from A to B in backward direction, through the infinite point. The arc $\mathcal{A}(\alpha)$ shares tangent with the spiral at the start point; so does $\mathcal{A}(-\beta)$ at the endpoint.

a, b are normalized (dimensionless) signed curvatures at the endpoints A, B of the arc.

$m = \operatorname{sgn}(b - a) = \pm 1$ is used to specify the kind of monotonicity, i.e. increasing or decreasing curvature.

α_0, β_0 are the angles of tangents to a curve, measured with respect to the x-axis. We index angular variables with 0 to signify boundary angles of short spirals (Kurnosenko, 2010a, St. 3):

$$\begin{aligned} &\text{increasing curvature } (m = +1)\colon \quad -\pi < \alpha_0 \leqslant \pi, \quad -\pi < \beta_0 \leqslant \pi; \\ &\text{decreasing curvature } (m = -1)\colon \quad -\pi \leqslant \alpha_0 < \pi, \quad -\pi \leqslant \beta_0 < \pi. \end{aligned} \tag{1}$$

ω_0, γ_0 are auxiliary variables, $\omega_0 = \frac{\alpha_0+\beta_0}{2}$, $\gamma_0 = \frac{\alpha_0-\beta_0}{2}$.

α, β denote cumulative boundary angles (11) such that $\alpha \equiv \alpha_0 \pmod{2\pi}$, $\beta \equiv \beta_0 \pmod{2\pi}$. For short spirals $\alpha = \alpha_0$, $\beta = \beta_0$.

σ, ω, γ: the signed angular width σ of the lens, its halfwidth ω, and the direction γ of the lens *bisector* $\mathcal{A}(\gamma)$ at the startpoint are

$$\sigma = \alpha + \beta, \qquad \omega = \frac{\alpha + \beta}{2}, \qquad \gamma = \frac{\alpha - \beta}{2}.$$

Generalized Vogt's theorem (Kurnosenko, 2011b, Th. 1) states that the sum $\alpha + \beta$ corresponds to the kind of spirality:

$$\operatorname{sgn}(b - a) = \operatorname{sgn}(\alpha + \beta), \quad \text{or} \quad m = \operatorname{sgn}\sigma. \tag{2}$$