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Two C^1-methods to generate Bézier surfaces from the boundary

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ABSTRACT

Two methods to generate tensor-product Bézier surface patches from their boundary curves and with tangent conditions along them are presented. The first one is based on the tetraharmonic equation: we show the existence and uniqueness of the solution of $\Delta^4\vec{\mathbf{x}} = 0$ with prescribed boundary and adjacent to the boundary control points of a $n \times n$ Bézier surface. The second one is based on the nonhomogeneous biharmonic equation $\Delta^2\vec{\mathbf{x}} = p$, where p could be understood as a vectorial load adapted to the C^1-boundary conditions.

1. Introduction

The aim of this work is to develop C^1-boundary based intuitive surface design techniques for Bézier surfaces which are one of the basic types of surfaces widely used in CAGD. The main idea is to find polynomial solutions to some natural PDEs which can only be controlled through the boundary control points and those adjacent to them. The two PDEs used are based on the Laplacian operator, so the solutions can be seen as extremals of the corresponding energy functionals.

According to previous results in Monterde (2004), given two opposed boundaries of a Bézier surface there is a unique harmonic surface ($\Delta\vec{\mathbf{x}} = 0$) with that prescribed partial boundary. And given the four opposed boundaries of a Bézier surface there is a unique biharmonic surface ($\Delta^2\vec{\mathbf{x}} = 0$) with that prescribed boundary (see Monterde and Ugail, 2004; Jüttler et al., 2006).

This last result allows us to give a simple method to generate a Bézier surface from its boundary. In this paper we study a similar problem, when not only the boundaries are prescribed but also the derivatives along them.

The idea behind the first method is just, after having seen what happens with harmonic and biharmonic surfaces, to increase the order of the partial differential operator. It is reasonable to think that if the harmonic condition $\Delta\vec{\mathbf{x}} = 0$ completely determines the Bézier surface $\vec{\mathbf{x}}$ from just two opposed boundaries, and if the biharmonic condition $\Delta^2\vec{\mathbf{x}} = 0$ completely determines the Bézier surface from the whole boundary, then the tetraharmonic condition $\Delta^4\vec{\mathbf{x}} = 0$ should completely determine the Bézier surface from the boundaries and the derivatives along the boundaries.

In Appendix A it is proved that given the boundary control points of a Bézier surface and those adjacent to them in the control net, then there exists a unique tetraharmonic polynomial surface satisfying such boundary conditions. Moreover, an explicit algorithm to compute the solution is given. These C^1 conditions on the boundary enable us to control the shape of

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the surface near these boundary, which can be very useful in a variety of different situations such as engineering or even virtual design.

The second method is based on a modification of the biharmonic condition. If just with the four boundary curves, a unique biharmonic Bézier surface is determined, then, in order to manage with the conditions related to the derivatives along the boundaries, one has to introduce more degrees of freedom. One possibility is to substitute the homogeneous biharmonic equation $\Delta^2\vec{\mathbf{x}} = 0$ by the nonhomogeneous one $\Delta^2\vec{\mathbf{x}} = \vec{\mathbf{y}}$.

The scalar nonhomogeneous biharmonic equation $\Delta^2 f = p$ describes the deflection of $f(u, v)$ of the middle surface of an elastic isotropic flat plate of uniform thickness and where $p(u, v)$ is the load per unit area, the coordinates u, v being taken in the plane $z = 0$ of the middle surface of the plate before bending. See Meleshko (1998) for a complete study of the biharmonic problem in a rectangle. The homogeneous biharmonic equation can be understood as a thin plate problem without load.

The new degrees of freedom we need will be under the form of an ad hoc vectorial load, which we will choose mainly concentrated along the boundaries. Intuitively, the reasoning could be the following: the prescription of the boundaries completely determines a biharmonic Bézier surface, $\vec{\mathbf{x}}_0$, which, in general, would not satisfy the conditions related with the derivatives along the boundaries. Therefore, the introduction of a load, mainly concentrated along the boundaries, acting on $\vec{\mathbf{x}}_0$ would bend the surface up to verifying the derivative conditions.

The use of *thin plate* methods in CAGD is well known from the very starting days of the subject. For example, one has to recall the notion of TP splines (see Farin, 2002). In this work such guiding principles are particularized to polynomial solutions. The idea of adding a new term to the load in the nonhomogeneous biharmonic equation to obtain polynomial approximations is not new either. It dates back to Biezeno and Koch (see Meleshko et al., 2001) and it can be said that the new load added by these two authors was also a polynomial load.[1]

Moreover, in the recent paper (Bloor and Wilson, 2006) the authors make use of polynomials solutions of the biharmonic equation as a first approximation to the true solution when the boundaries are not necessarily polynomial.

In Section 6 we compare our results with those in Bloor and Wilson (2006) which are also compared to Timoshenko's results. The major difference is that Bloor and Wilson's final approximation has a nonpolynomial term added while our final solution is polynomial, which is useful for computational purposes.

Let us say too that both methods are related by the following argument: any solution of the tetrahamonic equation $\Delta^4\vec{\mathbf{x}} = 0$ can be seen as a solution of the nonhomogeneous biharmonic equation $\Delta^2\vec{\mathbf{x}} = p$ with p a biharmonic load, this is, with p such that $\Delta^2 p = 0$. Therefore, in the first method we look for solutions of the nonhomogeneous biharmonic equation with a biharmonic load whereas in the second method we look for solutions with a load mainly concentrated along the boundary.

For the sake of clarity all the proofs of the main theorems as well as all the lemmas needed for these proofs have been put at the end of the paper in Appendixes A and B.

Finally the authors would like to thank the referees for all their useful comments which have helped to make the paper easier to follow and understand.

2. Background on biharmonic Bézier surfaces

The usual statement of the biharmonic problem as can be seen in (Meleshko, 1998) involves the prescription of the boundaries and of the normal derivatives along the boundaries. The surprising fact is that if we are looking for polynomial solutions of the homogeneous biharmonic equation, then we only need to prescribe a polynomial boundary in order to uniquely determine a solution.

Proposition 1. *Let $\vec{\mathbf{x}}_n(u, v) = \sum_{k,\ell=0}^{n} B_k^n(u)B_\ell^n(v)P_{k\ell}$ be a biharmonic Bézier chart of degree n with control net $\{P_{k\ell}\}_{k,\ell=0}^{n}$. Then all the inner control points $\{P_{k\ell}\}_{k=1,\ell=1}^{n-1}$ are determined by the boundary control points, $\{P_{0\ell}\}_{\ell=0}^{n}$, $\{P_{n\ell}\}_{\ell=0}^{n}$, $\{P_{k0}\}_{k=0}^{n}$ and $\{P_{kn}\}_{k=0}^{n}$.*

Remark 1. The boundary control points are

$$\begin{array}{cccccc} P_{00} & P_{01} & P_{02} & \dots & P_{0n-1} & P_{0n} \\ P_{10} & * & * & \dots & * & P_{1n} \\ P_{20} & * & * & \dots & * & P_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-1,0} & * & * & \dots & * & P_{n-1,n} \\ P_{n0} & P_{n1} & P_{n2} & \dots & P_{n,n-1} & P_{nn} \end{array}$$

The proof of Proposition 1 can be seen in Monterde and Ugail (2004), moreover, in Monterde and Ugail (2006) there is the detailed algorithm which allows to compute, given the boundary, the unique polynomial solution of any 4-th order

[1] The new term added to the load has been called in Meleshko et al. (2001) *fictitious load*. In our situation, where there is not a real load from the beginning, we will not use the adjective fictitious.