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Technical section

Convexity control of a bivariate rational interpolating spline surfaces

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Abstract

A bivariate rational interpolation surface based on function values has been constructed in the authors' earlier works. This paper deals with the convexity control of interpolating surfaces. The sufficient and necessary conditions for interpolating surfaces to be convex are derived. The convexity of the interpolating surface can be changed locally by selecting suitable parameters under the condition that the interpolation data are not changed. Examples are given to show how the parameters can be chosen and the shapes of the surfaces changed.

Keywords: Convexity control; Bivariate interpolation; Rational spline

1. Introduction

The construction methods of curves and surfaces are key issues in computer-aided geometric design (CAGD). There are many ways to tackle this problem [1–6], for example, the spline method, the Non-Uniform Rational B-Spline (NURBS) method, the Bézier method and others. These methods are effective and applied widely in the shape design of industrial products. In order to meet the needs of the ever-increasing model complexity and to incorporate manufacturing requirements, shape control becomes an ever more important task in constructing curves and surfaces, such as convexity control, positivity control, monotone control, etc.

Spline interpolation is a useful and powerful tool in CAGD, such as the polynomial spline, the triangular spline, the β-spline, the Box spline, the vertex spline and others [1–3,6]. There are many publications contributing to the shape preserving property of interpolating curve and surface [10–19], but only a few methods for shape control [7–9]. This is because of the global property of the interpolation, which means if impossible for a local modification to take place under the condition that the given interpolating data are not changed. In fact, there are some methods for preserving positivity or preserving convexity in the design of surfaces [11,12,14,15], but there are few modification methods to control the shape of the interpolating surface under the condition that the interpolating data are not changed [20,21]. This is because of the uniqueness of the interpolation to the interpolating data. In recent years, the univariate rational spline interpolation with parameters has been constructed [8,9,17–19,22]. These kinds of interpolation spline have a simple mathematical representation, and they can be used not only for shape preserving [3,20,21,23], but also for the modification of local curves, such as region control and convexity control [8,9,22], by selecting suitable parameters under the condition that the interpolating data are not changed. In this case, the uniqueness of the interpolating curves for the given interpolating data becomes the uniqueness of the interpolating curves for the given interpolating data and the parameters.

Motivated by the univariate rational spline interpolation, bivariate rational interpolation with parameters based on the function values has been studied in [20]. The interpolation function has a piecewise explicit rational mathematical representation with parameters, and it can be represented by its basis. Since there are parameters in the interpolation function, the interpolating surface varies as

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the parameters change. So, the variation of the parameters makes the modification of the interpolating surface possible under the condition that the interpolation data are not changed. This is the main advantage of the rational spline with parameters. For example, when a patch of the interpolating surface is too high or too low at a given point and at its neighbourhood, by adjusting these parameters, the surface can be constrained to be "down" or "up", so the shape of the interpolating surface can be modified to the desired shape. Ref. [21] gives such a method, a point control method, for the "down-up" control of interpolating surfaces. This paper will deal with the convexity control method to modify the convexity of the interpolating surface based on the definition of the surface's convexity defined by the Gauss curvature.

This paper is arranged as follows. In Section 2, the bivariate rational interpolation based on the function values will be restated. Section 3 deals with the convexity control of interpolating surfaces. The sufficient and necessary conditions for the interpolating surfaces to be convex are derived. Based on this, suitable parameters can be chosen automatically to ensure the interpolating patch is convex. In Section 4, examples are given to show this control method.

2. Interpolation

Let $\Omega : [a, b; c, d]$ be the plane region, let $f(x, y)$ be a bivariate function defined in the region Ω and let $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$ and $c = y_1 < y_2 < \cdots < y_m < y_{m+1} = d$ be the knot sequences. Denote $f(x_i, y_j)$ by $f_{i,j}$, then $\{(x_i, y_j, f_{i,j}), i = 1, 2, \ldots, n, n+1; j = 1, 2, \ldots, m, m+1\}$ are the given set of data points. For any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, let $h_i = x_{i+1} - x_i$, $\theta = (x - x_i)/h_i$, and $l_j = y_{j+1} - y_j$, $\eta = (y - y_j)/l_j$. For each $y = y_j$, $j = 1, 2, \ldots, m+1$, construct the x-direction interpolant curve [8]; this is given by

$$P^*_{i,j}(x) = \frac{p^*_{i,j}(x)}{q^*_{i,j}(x)}, \quad x \in [x_i, x_{i+1}], \;\; i = 1, 2, \ldots n-1, \tag{1}$$

where

$$p^*_{i,j}(x) = (1-\theta)^3 \alpha_{i,j} f_{i,j} + \theta(1-\theta)^2 V^*_{i,j} + \theta^2(1-\theta) W^*_{i,j} + \theta^3 f_{i+1,j},$$

$$q^*_{i,j}(x) = (1-\theta)\alpha_{i,j} + \theta,$$

and

$$V^*_{i,j} = (\alpha_{i,j} + 1) f_{i,j} + \alpha_{i,j} f_{i+1,j},$$

$$W^*_{i,j} = (\alpha_{i,j} + 2) f_{i+1,j} - h_i \Delta^*_{i+1,j},$$

with $\alpha_{i,j} > 0$, and $\Delta^*_{i,j} = (f_{i+1,j} - f_{i,j})/h_i$. This interpolation is called the rational cubic interpolation based on function values which satisfies

$$p^*_{i,j}(x_i) = f_{i,j}, p^*_{i,j}(x_{i+1}) = f_{i+1,j}, p^{*'}_{i,j}(x_i) = \Delta^*_{i,j}, p^{*'}_{i,j}(x_{i+1}) = \Delta^*_{i+1,j}.$$

Obviously, the interpolation is a local one, it is defined in the interval $[x_i, x_{i+1}]$ and depends on the data at three points $\{(x_r, y_j, f_{r,j}), r = i, i+1, i+2\}$ and the parameter $\alpha_{i,j}$.

For each pair $(i,j), i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, m-1$, using the x-direction interpolation function $P^*_{i,j}(x)$, define the bivariate rational interpolating function $P_{i,j}(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows [20]:

$$P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \quad i = 1, 2, \ldots n-1; \;\; j = 1, 2, \ldots, m-1, \tag{2}$$

where

$$p_{i,j}(x, y) = (1-\eta)^3 \beta_{i,j} P^*_{i,j}(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta) W_{i,j} + \eta^3 P^*_{i,j+1}(x),$$

$$q_{i,j}(y) = (1-\eta)\beta_{i,j} + \eta,$$

and

$$V_{i,j} = (\beta_{i,j} + 1) P^*_{i,j}(x) + \beta_{i,j} P^*_{i,j+1}(x),$$

$$W_{i,j} = (\beta_{i,j} + 2) P^*_{i,j+1}(x) - l_j \Delta_{i,j+1}(x),$$

with $\beta_{i,j} > 0$, and $\Delta_{i,j}(x) = (P^*_{i,j+1}(x) - P^*_{i,j}(x))/l_j$.

The function $P_{i,j}(x, y)$ is defined in the subregion $[x_i, x_{i+1}; y_j, y_{j+1}]$ and depends on the data at nine points $\{(x_r, y_s, f_{r,s}), r = i, i+1, i+2, s = j, j+1, j+2\}$; it is called the bivariate rational interpolating function based on function values which satisfy

$$P_{i,j}(x_r, y_s) = f(x_r, y_s), r = i, i+1, s = j, j+1.$$

It was proved in [20] that when the parameters $\beta_{i,j}$ is constant, for each $j \in \{1, 2, \ldots, m-1\}$ and all $i = 1, 2, \ldots, n-1$, the interpolating function $P_{i,j}(x, y), i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ must be C^1 continuous in the whole interpolating region $[x_1, x_n; y_1, y_m]$. In what follows, consider the equally spaced knots case, namely, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, $h_i = h_j$ and $l_i = l_j$. Assume $\beta_{i,j}$ is constant for each $j \in \{1, 2, \ldots, m-1\}$ and all $i = 1, 2, \ldots, n-1$; denote it by β_j. Assume $\alpha_{i,j}$ is constant for each $i \in \{1, 2, \ldots, n-1\}$ and all $j = 1, 2, \ldots, m-1$, and denote it by α_i. Under the conditions above, the interpolating function $P_{i,j}(x, y)$ is C^1 continuous in the whole interpolating region, and the interpolating function $P^*_{i,j}(x)$ defined by (1) can be rewritten as

$$P^*_{i,j}(x) = \omega_0(\theta, \alpha_i) f_{i,j} + \omega_1(\theta, \alpha_i) f_{i+1,j} + \omega_2(\theta, \alpha_i) f_{i+2,j},$$

where

$$\omega_0(\theta, \alpha_i) = \frac{(1-\theta)^2(\alpha_i + \theta)}{(1-\theta)\alpha_i + \theta},$$

$$\omega_1(\theta, \alpha_i) = \frac{\theta(1-\theta)\alpha_i + 3\theta^2 - 2\theta^3}{(1-\theta)\alpha_i + \theta},$$

$$\omega_2(\theta, \alpha_i) = \frac{-\theta^2(1-\theta)}{(1-\theta)\alpha_i + \theta},$$

and

$$\sum_{r=0}^{2} \omega_r(\theta, \alpha_i) = 1.$$