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Chaos and Graphics

Reverse bifurcations in a unimodal queueing model

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Abstract

We present a family of unimodal maps, arising from a simple queueing model, which exhibits reverse bifurcations. We compare and contrast this with bifurcations occurring in the well-known logistic family of unimodal quadratic maps. Throughout this study, graphics generated via numerical simulations provide key insights.

Keywords: Period-doubling, tangent, and reverse bifurcation; Unimodal map; Queueing system

1. Introduction

In a recent paper in this journal [1], Frame and Meachem presented computer experiment graphics in which a quartic family of one-dimensional maps exhibited reverse bifurcations. In particular, tangent and period-doubling bifurcations, well-known to occur in the logistic family [2], appear in this quartic family with both forward and reverse orientations. In this paper we present a unimodal (or one-hump) one-dimensional map, derived from a simple queueing model, which exhibits bifurcations akin to those found in the study of the quartic family. Interestingly, reverse bifurcations occur in mappings much less topologically sophisticated than a three-hump quartic function.

2. Bifurcations

The road to chaos for one-dimensional maps often follows a sequence of period-doubling bifurcations as a parameter increases, as illustrated by the logistic family $L_k(x) = kx(1-x)$. It is easy to check that if $k \in [0, 4]$, then for any $x_0 \in [0, 1]$, $L_k(x_0) \in [0, 1]$. For such k, the *orbit* of x_0, defined to be the set $\mathcal{O}^+(x_0) = \{x_0, x_1 = L_k(x_0), x_2 = L_k(x_1), \ldots\}$, remains bounded within [0, 1]. Of interest is how orbits $\mathcal{O}^+(x_0)$ change as the parameter k increases.

The simplest way to proceed is via the study of graphics generated by computer simulation. Indeed, the use of such graphics has played, and continues to play, a major role in the study of iteration of families of mappings. In Fig. 1 we plot the long-term behavior of $\mathcal{O}^+(0.5)$ versus k. That is, for a given k, we compute the first 200 terms in $\mathcal{O}^+(0.5)$, then plot iterates 101–200 vertically above k. The resulting plot is called an *orbit diagram*. The choice of $x_0 = 0.5$ is due to a result of Singer [3], which states that for a map having negative Schwarzian derivative, any attracting periodic behavior must draw in the orbit of a critical point. One can show L_k has negative Schwarzian derivative, with $x_0 = 0.5$ its sole critical point.

As this remarkable diagram is discussed at length in [1], we focus here on the two types of bifurcations which occur within. Note that for k slightly less than 3, $\mathcal{O}^+(0.5)$ converges to a stable fixed point (depending on k). A *fixed point* is a point of intersection of the graph of L_k and the line $y = x$; it is *stable* if it attracts orbits which start nearby. As proven in [2], a fixed point $x = p$ is stable if $|L_k'(p)| < 1$. A *period n-point* $x = q$ is a fixed point of the n-fold composition $L_k^n(x)$, and it is stable if $|(L_k^n)'(q)| < 1$. The orbit of a period n-point is called an *n-cycle*. An orbit which starts sufficiently close to any point in a stable n-cycle will limit on that period-n behavior over time.

For k slightly larger than 3, $\mathcal{O}^+(0.5)$ converges to a stable 2-cycle. This *period-doubling bifurcation* as k increases through 3 is illustrated in Fig. 2. Note that the fixed point

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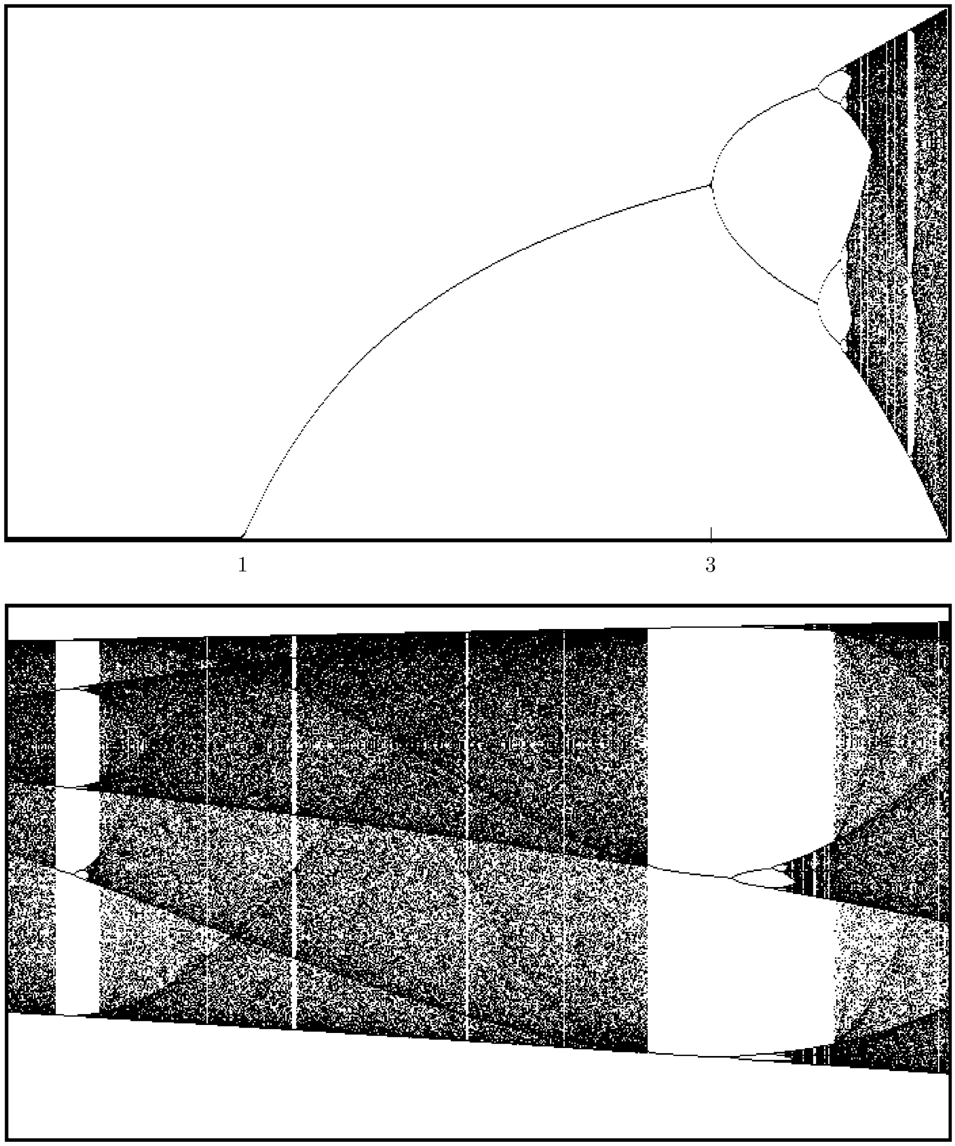

Fig. 1. Top: The orbit diagram for the logistic map. Bottom: The portion of the orbit diagram for $3.72 \leqslant k \leqslant 3.88$. Note the period-5 and period-3 "windows".

$x = p_k$ changes from stable to *unstable* (that is, $|L'_k(p_k)| > 1$ so that orbits which start nearby move away). As k continues to increase, a stable 4-cycle appears via a period-doubling bifurcation of the function $L^2_k(x)$, followed by a stable 8-cycle arising in a period-doubling bifurcation of the function $L^4_k(x)$, and so on. These period-doublings continue until $k \approx 3.57$, where chaos first appears.

The second type of bifurcation occurring in the logistic family is a *tangent bifurcation*. Note the 3-cycle appearing out of the chaotic morass in Fig. 1. The function $L^3_k(x)$ undergoes a tangent bifurcation at $k \approx 3.828$, as illustrated in Fig. 3. Thus, L_k progresses from having no 3-cycles, to one 3-cycle, and then to a stable–unstable pair of 3-cycles as k increases through 3.828.

It is quite fun investigating the plot in Fig. 1 by zooming in on various regions—it is indeed a surprisingly rich diagram. Regardless of how you search, however, you will find only period-doubling and tangent bifurcations. In addition, once a cycle is created it persists, though it changes from stable to unstable as discussed above. That this is the case follows from work of Milnor and Thurston [4]. That this need not be the case for quartic maps is the substance of [1]. In the following we present a unimodal map with period-doubling, tangent and reverse bifurcations similar to those found in the quartic family. Here, unimodal means that the map has one critical point c, and is either increasing for $x < c$ and decreasing for $x > c$, or vice versa. Note that while all quadratic maps are unimodal, a unimodal map need not be quadratic.

Interestingly, our unimodal map arises in a simple queueing model [5].