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Deformation simulation using cubic finite elements and efficient *p*-multigrid methods

Daniel Weber [a,b,*], Johannes Mueller-Roemer [a,b], Christian Altenhofen [a,b], André Stork [a,b], Dieter Fellner [a,b,c]

[a] Fraunhofer IGD, Darmstadt, Germany
[b] GRIS, TU Darmstadt, Germany
[c] CGV, TU Graz, Austria

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ABSTRACT

We present a novel *p*-multigrid method for efficient simulation of corotational elasticity with higher-order finite elements. In contrast to other multigrid methods proposed for volumetric deformation, the resolution hierarchy is realized by varying polynomial degrees on a tetrahedral mesh. The multigrid approach can be either used as a direct method or as a preconditioner for a conjugate gradient algorithm. We demonstrate the efficiency of our approach and compare it to commonly used direct sparse solvers and preconditioned conjugate gradient methods. As the polynomial representation is defined w.r.t. the same mesh, the update of the matrix hierarchy necessary for corotational elasticity can be computed efficiently. We introduce the use of cubic finite elements for volumetric deformation and investigate different combinations of polynomial degrees for the hierarchy. We analyze the applicability of cubic finite elements for deformation simulation by comparing analytical results in a static and dynamic scenario and demonstrate our algorithm in dynamic simulations with quadratic and cubic elements. Applying our method to quadratic and cubic finite elements results in a speed-up of up to a factor of 7 for solving the linear system.

1. Introduction

In the realm of computer graphics several classes of algorithms have been proposed for simulating volumetric deformation. These include, among others, position-based dynamics [1], mass–spring systems and finite element methods [2] (FEM). FEM is a popular choice when intuitive material parameters and accuracy are important, as these methods are based on continuum mechanics and do not require parameter tuning to achieve physically plausible results. Instead of the commonly used linear basis functions, some authors propose to simulate volumetric deformation on tetrahedral meshes using quadratic finite elements, i.e., using polynomials of degree two (see, e.g., the work of Mezger et al. [3] or recently Bargteil et al. [4]). This leads to improved accuracy as higher polynomial degrees can better approximate the solution of the partial differential equation. Even though the number of degrees of freedom per element increases, the computational time can be reduced as the desired deformation can be represented with significantly fewer elements.

However, simulating deformation with finite elements is computationally expensive, as for a stable simulation, large, sparse linear systems must be solved in every time step. Typically, direct solvers like sparse Cholesky factorization or the method of conjugate gradients (CG) with preconditioning are chosen to solve these linear systems. These are usually the bottleneck of the simulation algorithm and become even more critical with increasing model sizes. A frequent compromise is to use a fixed number of CG iterations, but this in turn increases the numerical damping of the simulation and dissipates energy as we will demonstrate. In this paper we address this issue and propose a geometric *p*-multigrid method to efficiently and accurately solve sparse linear systems arising from higher order finite elements.

In our approach, we construct a hierarchy of varying degree polynomials to represent the field of unknowns. These are defined on the same tetrahedral discretization to iteratively improve the solution of the linear system. In contrast to other multigrid approaches for deformation simulation these levels vary in polynomial degree instead of mesh resolution. Furthermore, we introduce volumetric deformation simulation using cubic finite

* Corresponding author. Tel.: +49 6151 155202
E-mail address: daniel.weber@igd.fraunhofer.de (D. Weber).

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elements on tetrahedral meshes, which have not been proposed so far in the computer graphics community to the best of our knowledge. We represent the shape functions as polynomials in Bernstein–Bézier-form (B-form) and show how restriction and prolongation of shape functions can be incorporated into the polynomial hierarchy. Our contributions are as follows:

- We introduce a novel multigrid solver for volumetric deformation with higher order finite elements.
- We present deformation simulations with cubic finite elements in B-form.
- We show how polynomial representations in B-form of different degrees can be efficiently transformed into each other.
- We demonstrate a speed-up for higher order simulations up to a factor of 7 for solving the linear system in comparison to a preconditioned CG method.

This paper is based on our previous work [5] and extends it with some more detailed descriptions and analyses. More specifically, we describe the continuum mechanical approach and the finite element discretization in more detail. We also compare our purely geometric approach with Galerkin coarsening. Additionally, an algorithm using the *p*-multigrid method as a preconditioner for a conjugate gradient method is presented. Furthermore, we analyze the convergence of the multigrid-preconditioned CG solver (MGPCG) and the effect of approximating the field of rotations with a single rotation matrix only.

2. Related work

In the realm of computer graphics many methods for volumetric deformation simulation have been proposed. Most of the earlier work is summarized in the survey of Nealen et al. [6]. Besides methods based on solving the partial differential equations of linear, corotational or non-linear elasticity there are a number of other techniques for physically plausible volumetric deformation. In the early work of Baraff and Witkin [7] a numerical integrator for mass–spring systems and constraints was introduced allowing for arbitrarily large time steps. Although mass–spring systems have not been the focus of research for a long time, recently a block coordinate descent method was proposed by Liu et al. [8] where a constant stiffness matrix allows for efficient simulation. In the context of position-based dynamics a good summary of the relevant work is described in the survey of Bender et al. [1]. In contrast to these approaches we rely on continuum mechanical modeling of elasticity. This has the advantage that the simulation parameters such as, e.g., the material parameters have an intuitive meaning instead of parameters which require tuning depending on models and time step sizes.

Physically based simulation of deformation with FEM was first adopted by O'Brien and Hodgins [9]. They modeled and simulated brittle fracture using an explicit time stepping method. Mueller and Gross [2] used implicit time integration together with a corotational formulation. This allowed for stable simulations and avoided the artifacts of linear elasticity by recomputing the reference coordinate system. Irving et al. [10] presented a method to cope with inverted tetrahedral elements that occur when large forces are present. Parker and O'Brien [11] demonstrated how the corotational formulation for simulating deformation and fracture can be applied with strict computation time constraints in a gaming environment. Kaufmann et al. [12] used a discontinuous Galerkin method to simulate volumetric deformation. An extension of the corotational method that takes the rotational derivatives into account has been presented by Chao et al. [13], achieving energy conservation. In the context of deformable simulation with element inversion, Stomakhin et al. [14] propose an energy-based approach to handle inverted configurations more robustly. In order to allow for larger time steps without numerical damping Michels et al. [15] introduced exponential integrators for long-term stability due to energy conservation.

Simulation with higher-order finite elements is wide-spread in engineering applications (see e.g. [16]) where linear shape functions do not provide sufficient accuracy. In the computer graphics community quadratic finite elements were used for deformation simulation [3] and interactive shape editing [17]. In a previous work [18] we used quadratic finite elements in B-form for the simulation of volumetric deformation. Later we developed a GPU implementation for linear and quadratic finite elements [19]. Recently, Bargteil and Cohen [4] presented a framework for adaptive finite elements with linear and quadratic B-form polynomials. Our method builds upon the work of Bargteil and Cohen and extends it by additionally introducing cubic finite elements and employing an efficient method for solving the governing linear systems.

Multigrid methods in general have been the subject of extensive research. Standard textbooks like [20,21] provide a good overview of the basic method and its theory. Geometric multigrid methods are especially suited for discretizations on regular grids. In the context of deformation simulation Zhu et al. [22] propose a multigrid framework based on finite differences. Dick et al. [23] use hexahedral finite elements discretization on a regular grid and solve the linear systems using a GPU-based multigrid method. In [24] they extend this approach for simulating cuts. In contrast, our method employs a discretization on tetrahedral meshes, which allow for an adaptive approximation of the simulated geometry potentially requiring less elements.

Based on tetrahedral meshes Georgii et al. [25] proposed a geometric multigrid algorithm based on linear finite elements using nested and non-nested mesh hierarchies. In general this geometric concept cannot be easily adapted for higher order finite elements. In their work the computational bottleneck is the matrix update, where sparse matrix–matrix products (SpMM) are required on every level to update the multigrid hierarchy. Later in [26] they specifically developed an optimized SpMM to address this bottleneck and report a speed-up of one order of magnitude. However, the matrix update is still as expensive as the time for applying the multigrid algorithm itself. In contrast, our *p*-multigrid method employs polynomial hierarchies on a common tetrahedral mesh. As the problem is directly discretized the expensive SpMM operations are avoided.

For two-dimensional analysis of elliptic boundary value problems, Shu et al. [27] introduced a *p*-multigrid for finite elements using a higher-order Lagrangian basis. To the best of our knowledge we are the first to introduce this concept in three dimensions, to employ polynomials in B-form and to solve equations of elasticity with this algorithm.

In comparison to our previous work [5], on which this article is based, we show how to use the *p*-multigrid method as a preconditioner for a conjugate gradient method and analyze its convergence and the effect of a constant rotation approximation.

3. Higher-order finite element discretization of elasticity

In this section we briefly outline the general approach for simulating volumetric deformation using higher order finite elements. First, we describe the general process of higher order discretization and then outline the steps necessary for corotational elasticity.