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Robust visualization of strange attractors using affine arithmetic

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Abstract

We propose the use of affine arithmetic in cell-mapping methods for the robust visualization of strange attractors and show that the resulting cellular approximations converge faster than those produced by cell-mapping methods based on classical interval arithmetic.

Keywords: Strange attractors; Cell mapping; Interval arithmetic; Affine arithmetic; Discrete dynamical systems

1. Introduction

The goal in the study of discrete dynamical systems is to understand the long-term behavior of the iterates of a map $f: \mathbf{R}^n \to \mathbf{R}^n$. We are interested in what happens to the *orbit* $\mathcal{O}(p)$ of a point $p \in \mathbf{R}^n$:

$$\mathcal{O}(p) = \{p, f(p), f(f(p)), f(f(f(p))), \ldots\}.$$

Typically, such orbits either diverge to infinity or converge to a manifold in $\mathbf{R}^n$. (A *manifold* is a well-behaved set, such as a point, a set of isolated points, a curve, a surface, etc.) This limit set is called the *attractor* of the dynamical system. Not all attractors are well behaved and in many cases the orbits accumulate on sets that have complicated geometry and topology. Such limit sets are known as *strange attractors* and can exist even for the simplest non-linear maps f. A prime example of this phenomenon is given the famous Hénon map [1], which acts on the plane $\mathbf{R}^2$ as follows:

$$f(x, y) = (1 + y - ax^2, bx),$$

where a and b are parameters. The Hénon strange attractor is obtained by setting $a = 1.4$ and $b = 0.3$; it is shown in Fig. 1. (Strictly speaking, it has not been mathematically proved that the Hénon attractor is actually a strange attractor in the technical sense. The mathematics of the Hénon map is very complicated and its dynamics is not yet fully understood [2].)

Mathematicians usually start their study of a dynamical system by drawing a picture of its orbits. The simplest method for producing such a picture is the point-sampling method discussed in Section 2. However, as also discussed in Section 2, this method is not robust: it depends on trial and error, is subject to rounding errors, and may produce pictures that are not reliable. Other methods for approximating attractors reliably have been proposed, such as the cell-mapping method, which we discuss in Section 3. Although it can be based on point sampling, the cell-mapping method is made robust by using interval arithmetic [3], as described by Michelucci [4]. We propose here the use of affine arithmetic [5] instead of interval arithmetic in cell mapping. In Section 4 we show some examples of the performance of the cell-mapping method based on affine arithmetic for creating robust pictures of strange attractors.

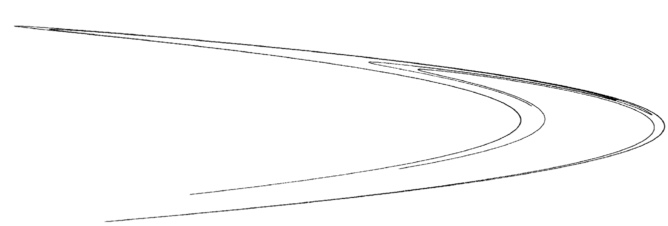

Fig. 1. The Hénon strange attractor.

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2. Point sampling

The simplest way to "see" the dynamics of a discrete system is to draw a picture of its orbits using the following sampling method:

(1) Guess or somehow find a box Ω containing the attractor.
(2) Select a set of random starting points in Ω.
(3) For each starting point $p \in \Omega$, compute but do not plot the first n_0 points in the orbit $\mathcal{O}(p)$.
(4) Compute and plot the next n_1 points in $\mathcal{O}(p)$.

If the attractor is a manifold, then picture will show a fairly dense approximate sampling of the manifold. If the orbits diverge to infinity, the picture will show nothing (if we do not plot outside Ω). For strange attractors, the picture will show a cloud of dots that clearly has some structure, but this structure is elusive to describe. For instance, detailed pictures of the Hénon attractor suggest that it has Cantor-set cross-sections [1] and thus has a fractal nature.

The point-sampling method is very simple to understand and implement. It generates nice pictures. The main difficulty we face when using this method is how to choose n_0 and n_1. Choosing n_0 too small will include transient parts of the orbits, that is, points that are not yet near the attractor. Choosing n_1 too small will risk not covering the attractor sufficiently well. On the other hand, choosing n_0 or n_1 too large may be wasteful and inefficient. In practice, we just choose n_0 and n_1 by trial and error. However, there is no way to be sure that we have selected good values for n_0 or n_1.

Another difficulty with the point-sampling method is that it is implemented using floating-point arithmetic, which is subject to rounding errors [6]. For chaotic dynamical systems—the ones that have strange attractors—rounding errors are potentially serious, because orbits starting at nearby points can diverge from each other exponentially. Sometimes, this strong sensitivity to initial conditions does not affect the overall picture, because numerically computed orbits are "shadowed" by exact orbits that capture the typical behavior of the system. However, the truth is that rounding errors affect numerical simulations of dynamical systems in very complex ways [7]. Well-conditioned dynamical systems may display chaotic numerical behavior [8,9]. Conversely, numerical methods can suppress chaos in some chaotic dynamical systems [9].

As a consequence of both difficulties, the pictures generated with the point-sampling method will probably not display the attractor reliably. This is specially serious when we have just started investigating a dynamical system and its attractor is not yet known.

3. Cell mapping

An alternative to point sampling is cell mapping [10,11]. The main idea in this method is to decompose Ω into cells (typically using a uniform rectangular grid) and study the dynamics induced by f on this set of cells. Instead of asking where each point goes under f, we ask where each cell goes. More precisely, we consider the directed graph having the cells as vertices and having an edge from cell A to cell B if $f(A)$ intersects B. This means that A goes (partially) to B. This graph is called the *cell graph*. The key observation in the cell-mapping method is that the strongly connected components of the cell graph must cover the attractor of f. Cells having no edge into them cannot contain any part of the attractor because f never takes points into those cells. Cells in the same strongly connected component are (partially) mapped into each other by iterates of f. Thus, strongly connected components capture the *transitivity* of the attractor. Fig. 2 shows a cell graph for the Hénon map.

Given a sufficiently fine cell decomposition of Ω, we can find a good approximation of the attractor by finding the strongly connected components of the corresponding cell graph. A more efficient approximation for the attractor can be found by using recursive subdivision: start with a coarse cell decomposition of Ω; find the cell graph induced by f; find its strongly connected components; subdivide the cells in these components into smaller cells; rebuild the cell graph using the smaller cells; find its strongly connected components; and repeat until the cells are small enough. Efficiency comes from not having to start from a very fine cell decomposition; only the cells in the strongly connected components are refined. Fig. 3 shows the convergence of this recursive subdivision to the Hénon attractor with classical parameters $a = 1.4$ and $b = 0.3$.

Finding the strongly connected components of a graph can be done in time linear in the size of the graph, using an elegant algorithm by Tarjan [12]. This leaves as the main difficulty in the cell-mapping method how to find the edges in the cell graph, that is, how to decide which cells $f(A)$ intersect. We call this the *edge problem*. Because f is a non-linear map, there is no simple

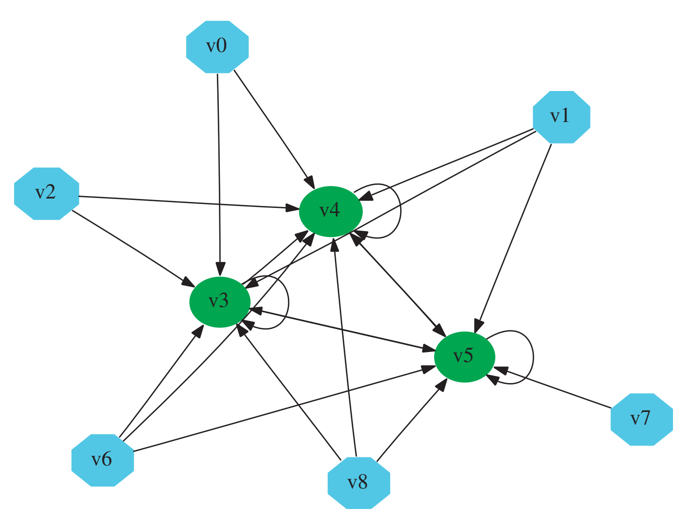

Fig. 2. Cell graph of the Hénon map based on a 3×3 rectangular subdivision. The strongly connected component is shown in green.