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Invariances of single curved manifolds applied to mesh segmentation

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ABSTRACT

Recently, it has been shown that for the Lambert illumination model, only scenes composed of developable objects with a very particular albedo distribution produce an (2D) image with isolines that are (almost) invariant to light direction change.

In this work, we provide and investigate a more general framework. We show that, in general, the requirement for such invariances is quite strong and is related to the differential geometry of the objects. More precisely, we prove that single-curved manifolds, i.e., manifolds such that at each point, there is at most one principal curvature direction, produce invariant isosurfaces for a certain relevant family of energy functions. In the three-dimensional case, the associated energy function corresponds to the classical Lambert illumination model with albedo. This result is also extended for finite-dimensional scenes composed of single-curved objects. A direct consequence is that the Weiss et al. image change detection algorithm (On the illumination invariance of the level lines under directed light: Application to change detection. SIAM J Imag Sci 2011;4(1):448–71) can be extended to detect whether a manifold is single curved or not, in any finite dimension. We design a mesh segmentation procedure based on such a result, and we implement the procedure in 3D.

1. Introduction

The broad range of *nuisance factors*, e.g., noises and measurement failures, makes pattern recognition a hard task to automatize because such factors exhibit *variability*, which leads to an explosion of different patterns that need to be recognized. However, nuisance factors can sometimes be handled by extracting a *sufficient statistic* [24,32] of the input data that is invariant to the nuisance factors. This type of technique is popular in the computer vision community [2,12,28,31,34,33]. The measurements are often a two-dimensional image of a three-dimensional scene; the nuisance factors include illumination conditions, viewpoint changes, occlusions, shadows, quantizations, and noises.

Related work: To extract sufficient statistics from the input data, it is necessary to consider a mathematical model of the data source. With this in mind, it is worthwhile mentioning that excellent models have been produced by the scientific community to reproduce the effects of illumination on artificial scenes; see [20] for a comprehensive description. Perhaps the most well-known model is the Lambert illumination model [29]. There are a number of interesting statistics for image data. We can cite the *contours* [26], the *topographic map* [6,8], or the *attributed Reeb tree* [31] of the image, to name a few. The topographic map, which is the set of all *isolines* of the image, has some advantages: (i) it is contrast-invariant [2]; (ii) it allows for image reconstruction, while not depending on any thresholding parameter; and (iii) several problems have been successfully tackled by using topographic maps [27,4,35].

To the best of our knowledge, it has been shown that for *Lambertian objects*, it is always possible to construct nontrivial viewpoint-invariant image statistics [33], whereas general-case (global) illumination invariants essentially do not exist at all [12]. There is a body of literature dealing with more restricted illumination models [2,27,3], but for the Lambertian model, necessary and sufficient conditions for the scene geometry to have a topographic map of the image that is (almost) invariant to the incident light direction have only recently been found [34]. More precisely, it has been proved that only scenes composed of *developable objects* with a very particular *albedo* distribution produce an image with *isolines* that are (almost) invariant to light direction change.

Surprisingly, a generalization of such an invariant, which is given in this paper, can be applied in mesh segmentation as well. Some segmentation algorithms in 3D may benefit from good initial guesses [21,22]. When the segmentation process is related to curvature estimates of the surface, then a reasonable initial guess

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might be segments of the surface that are "more likely" to be developable [13], i.e. having at most one curvature direction at each point. One could detect segments of the surface that are likely to be developable by simply estimating discrete curvatures, which can be done fairly well (e.g., osculating jets [10] or normal cycles [14]), and running a clustering algorithm. However, these estimators are not optimized to detect surface developability but rather to obtain precise estimates of curvatures in general, which is a harder task. As a consequence, they are quite slow. In this paper, we are able to design a fast algorithm that identifies segments of the surface that are likely to be developable without any need for curvature estimation.

Contributions: This is a significantly extended and revised version of [9], where we extended the theoretical work of Weiss et al. [34] and designed an algorithm to detect segments of manifolds with a Gauss map rank of 1 at most. The following contributions are specific to this paper: (i) we give more detailed results of our algorithm, with more meshes and timings; (ii) we present a preprocessing step involving a Lloyd-type optimization procedure, which gives a better segmentation result; and (iii) we analyze the robustness of our algorithm.

Paper overview: In Section 2, we show that under some conditions, if a collection of objects is composed of Riemannian manifolds with Gauss map ranks of 1 at most, then the isosurfaces of a certain energy function on these objects are *almost* invariant. In Section 3, we present a mesh segmentation algorithm to detect pieces of a surface that are likely to be developable. In Section 4, we run our algorithms for several input meshes of different sizes and showcase computation times; we also discuss the influence of the parameters of our method on the results. In Section 5, we describe a preprocessing step that improves our segmentation results without having a significant impact on the computation time. In Section 6, we discuss the main robustness limitations of our method. Finally, in Section 7, we conclude the paper.

2. Equivalence for single-curved manifolds

2.1. Preliminaries

In this paper, *isosurfaces* are defined as follows [23].

Definition 1 (*Isosurfaces*). Let $u : \Omega \to \mathbb{R}$ be some scalar field on a manifold Ω. The isosurfaces of u are defined as the connected components of $u^{-1}(\lambda)$, where λ is a constant.

In Section 2.2, we study the isosurfaces of a certain energy function defined on a manifold Ω_t, which depends on the gradient of a manifold Ω_s (mnemonics **s** and **t** correspond to the words **s**ource and **t**arget, respectively). The following definition clarifies this scenario.

Definition 2 (*Energy function*). We denote the classical scalar product between two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$, by $\langle \mathbf{x}, \mathbf{y} \rangle$, and the normal at $\mathbf{x}$ of some differentiable manifold Ω by $\vec{\mathbf{N}}_{\Omega}(\mathbf{x})$. Let Ω_s and Ω_t be *Riemannian manifolds* (smooth manifolds equipped with a Riemannian metric) with codimension 1, $\mathbf{v} = (v_1, \ldots, v_d, v_{d+1})$ be a vector in $\mathbb{R}^{d+1}$, $\alpha : \Omega_s \to \mathbb{R} - \{0\}$ be a scalar field, and $\psi : \Omega_s \to \Omega_t$ a diffeomorphism; then, the energy function $E_{\Omega_s,\mathbf{v}} : \Omega_t \to \mathbb{R}$ is defined as follows:

$$E_{\Omega_s,\mathbf{v}}(\mathbf{x}) = (\langle \mathbf{h}, \vec{\mathbf{N}}_{\Omega_s}(\psi^{-1}(\mathbf{x}))\rangle + \kappa) \cdot \alpha(\psi^{-1}(\mathbf{x})), \tag{1}$$

where $\mathbf{h} = (v_1, \ldots, v_d)$, and $\kappa = v_{d+1}$, and $d \geq 2$.

It is worthwhile noticing that in the third dimension, the scenario above corresponds to the Lambert illumination model, where the following holds: (i) a piece of the surface projected on

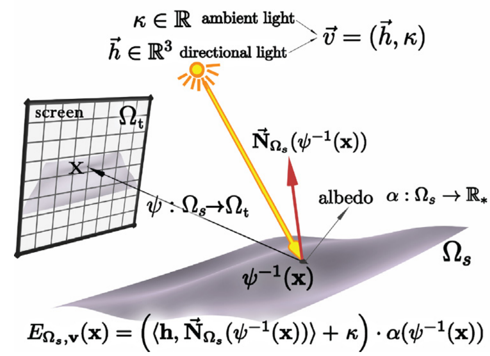

Fig. 1. *Lambert model*: For dimension three, the scenario in Definition 2 corresponds to the Lambert illumination model.

the screen is Ω_s, and the projection on the screen is Ω_t; (ii) ψ is the projection map itself; (iii) h and κ are the directional light vector and the ambient light constant of the scene; (iv) α is the albedo distribution of the surface; and (iv) the energy $E_{\Omega_s,\mathbf{v}}(\mathbf{x})$ is quantified to the actual gray level in the resulting image at $\mathbf{x}$. This scenario is depicted in Fig. 1.

However, in this work, our interest in $E_{\Omega_s,\mathbf{v}}$ is due to its isosurface's (almost) invariance to changes in the vector $\mathbf{v}$, which is proved in what follows. While being invariant to vector changes, the isosurfaces of $E_{\Omega_s,\mathbf{v}}$ carry enough information to discriminate developable patches from nondevelopable patches of the surface; this leads us to a fast segmentation algorithm, which is described in Section 3.

In the scenario above, we prove some properties when Ω_s is *single-curved* (or even composed of several single-curved objects). The following definition makes such a term more precise.

Definition 3 (*Single-curved manifolds*). We call a manifold Ω *single-curved* if and only if the rank of its *Gauss map* is everywhere less than or equal to 1.

In other words, Ω is single-curved if and only if it has at most one nonzero principal curvature direction at each of its points. Three-dimensional single-curved manifolds are simply (pieces of) developable surfaces; see Fig. 2. In higher dimensions, they are the *osculating scrolls* [19]. (At this point, the reader might want to refer to [15] for a comprehensive study of continuous differential geometry.)

2.2. Invariant isosurfaces

In this section, we show that under some conditions, the isosurfaces of $E_{\Omega_s,\mathbf{v}}$ in Ω_t are the same for almost all $\mathbf{v} \in \mathbb{R}^{d+1}$.

Theorem 4. *Given the conditions described in Definition2, if Ω_s is single-curved, and α varies only in the (nonzero) principal curvature direction of Ω_s (or freely if Ω_s has no curvature at all), then, for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, the isosurfaces of $E_{\Omega_s,\mathbf{v}_1}(\mathbf{x})$ are the same as those of $E_{\Omega_s,\mathbf{v}_2}(\mathbf{x})$.*

The proof of the theorem above can be found in the Appendix and is organized as follows: (i) we first prove an equivalence between Ω_s being single-curved and an invariance property on $\nabla E_{\Omega_s,\mathbf{v}}$; see Appendix A.1. Then, (ii) we prove that such an invariance property on $\nabla E_{\Omega_s,\mathbf{v}}$ implies that the isosurfaces of $E_{\Omega_s,\mathbf{v}}$ in Ω_t are the same for almost all $\mathbf{v} \in \mathbb{R}^{d+1}$; see Appendix A.2.

The *almost* expression in Theorem 4 is important because if the vectors are carefully chosen, then the isosurface might change; see Fig. 3. However, the set of such vectors is a null measurable set. Figs. 4 and 5 illustrate scenarios in which Theorem 4 can and cannot be applied, respectively.

Several objects can be modeled by pieces of single-curved (hyper-)surfaces; see Appendix A.3 for a precise definition. It turns