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Laplacians on flat line bundles over 3-manifolds

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ABSTRACT

The well-known Laplace–Beltrami operator, established as a basic tool in shape processing, builds on a long history of mathematical investigations that have induced several numerical models for computational purposes. However, the Laplace–Beltrami operator is only one special case of many possible generalizations that have been researched theoretically. Thereby it is natural to supplement some of those extensions with concrete computational frameworks. In this work we study a particularly interesting class of extended Laplacians acting on sections of flat line bundles over compact Riemannian manifolds. Numerical computations for these operators have recently been accomplished on two-dimensional surfaces. Using the notions of line bundles and differential forms, we follow up on that work giving a more general theoretical and computational account of the underlying ideas and their relationships. Building on this we describe how the modified Laplacians and the corresponding computations can be extended to three-dimensional Riemannian manifolds, yielding a method that is able to deal robustly with volumetric objects of intricate shape and topology. We investigate and visualize the two-dimensional zero sets of the first eigenfunctions of the modified Laplacians, yielding an approach for constructing characteristic well-behaving, particularly robust homology generators invariant under isometric deformation. The latter include nicely embedded Seifert surfaces and their non-orientable counterparts for knot complements.

1. Introduction and related work

From a physics and engineering perspective the well-known Laplacian acting on functions that are defined over some space M is essential for modeling common phenomena such as heat diffusion and wave propagation on M. In the corresponding mathematical models it often arises from variational methods applied to some energy minimization principle. Its properties make it a versatile tool for obtaining well-behaved functions or studying the underlying space.

The physical relevance and mathematical properties of the Laplacian have motivated several generalizations in various directions, such as the extension from scalar functions to vector or tensor fields. For example the vector Laplacian is relevant in the study of electromagnetics whereas analogous differential operators are used in linear elasticity. Furthermore, by going from Euclidean spaces to curved Riemannian spaces, the Laplace–Beltrami operator acting on functions and the Hodge–de Rham Laplacian acting on differential forms provide natural generalizations of the Laplacian or vector Laplacian, respectively. These and other more abstract generalizations are studied in a branch of mathematics known as spectral geometry.

Although many fundamental theoretical questions are still unsolved, the field of spectral geometry has established remarkable results that show the Laplacians to capture various geometric and topological information about the underlying space. However, most of these results are not directly amenable to computational methods and are rather given in terms of asymptotic relations or curvature-dependent bounds on the eigenvalues, see e.g. [1]. There is a large gap between the abstract constructions in theory and concrete computational methods applicable to given shapes. In particular, the transition from two to three dimensions is more challenging in practice than indicated by the general theory.

With the increasing availability of computing power, the Laplace operator has attracted considerable interest in computational geometry and shape processing, driven by the desire to exploit it for practical algorithms and based on a variety of discretizations, see e.g. [2–6]. Among applications employing numerically computed Laplacian invariants are shape and image retrieval using spectral prefixes [7–9] based on early research [10,11] and patented in [12] with a retrospective discussed in [13]. The Laplacian has also been successfully used in signal processing operations [14], surface remeshing [15], parametrization [16–18], mesh deformation [19], descriptors for shape matching [20–24],

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segmentation and registration [25] and statistical and topological shape analysis [26–28], just to mention a few. A survey of some of these and other applications can be found in [29].

The route taken in most of these applications is to start directly with a discretization defined in terms of concrete equations that are valid for a point cloud, graph, or mesh representation. Therefore, it is common to discuss the Laplace–Beltrami operator in a specific discretization, most notably as the so-called Cotangent–Laplacian [3]. However, comparatively few computational attempts have gone beyond modeling the classical Laplace–Beltrami operator by considering for example the spectrum of operators derived from different energy functionals [30,31], the Hodge–de Rham Laplacian [32] or quaternionic-valued operators [33].

The contribution of this paper fits into dealing with a class of operators going beyond the usual Laplace–Beltrami operator. We extend the work in [34,35], proposing a general method to explicitly construct a flat line bundle over a compact three-dimensional manifold M represented by a simplicial complex K and to perform a spectral decomposition for the associated connection Laplacian. Note that the general concept of connection Laplacians has recently also been investigated in the context of so-called vector diffusion maps for analyzing point-based data sets, see [36].

The case where M is equipped with non-Euclidean geometry and the trivial connection has been an object of study within physics, see for example [37,38] considering two-dimensional hyperbolic surfaces. Specific constantly curved three-dimensional settings have recently attracted attention, too, see e.g. [39]. Our method applies in these settings as well as in the general arbitrarily curved case.

As we will use knot complements to construct three-dimensional bounded manifolds, some of our result relate to so-called Seifert surfaces [40]. While these are topologically easily constructed, obtaining nice geometrical embeddings is challenging, see [41]. This topic has also been researched in the context of electromagnetic computations to deal with the multi-valuedness of scalar potentials by introducing cuts, see the work of Kotiuga [42,60]. As we will illustrate, our method yields well-behaved embeddings of Seifert surfaces or their non-orientable counterparts.

2. Contribution

Combining ideas from spectral geometry and algebraic topology, the aim of this paper is to investigate the so-called connection Laplacians on flat line bundles from a computational point of view. These operators generalize the well-known Laplace–Beltrami operator which has become ubiquitous in shape processing. One can interpret most of those Laplacians as perturbations of the ordinary Laplacian d^*d by a first-order differential expression, namely

$$\Delta_\omega f = d^*\, df + 2\langle df, \omega\rangle + (d^*\omega + |\omega|^2)f \tag{1}$$

where ω is an imaginary-valued closed differential one-form. Employing the notion of a connection, Δ_ω is often called the Bochner or connection Laplacian associated to the flat connection $d_\omega = d + \omega$. One way of understanding connection Laplacians is in terms of introducing certain sign flips or phase shifts across embedded hypersurfaces representing closed chains, i.e. so-called cycles within relative homology. Focusing on two-dimensional manifolds, an approach for obtaining the spectral decomposition of such Laplacians has been recently introduced in [34,35].

We follow up and extend those approaches by describing a general method that is able to deal with three-dimensional volumetric objects of complex topology. While in two dimensions it is comparatively easy to find a suitable 1-cycle resembling a curve and to perform the flips/phase shifts across this curve, the corresponding situation in three dimensions is more difficult. Obtaining suitable generators in this case requires more sophisticated algorithms which typically produce quite cluttered outputs. These generators can exhibit complex self-intersections or even be non-orientable, thereby obscuring how and where precisely to apply the required sign flips or phase shifts consistently.

In this paper we investigate topologically complex three-dimensional manifolds M by computing the spectral decompositions of the generalized Laplacians. Our approach applies to compact manifolds that may be unbordered and even equipped with a non-Euclidean geometry.

We describe how to overcome the above-mentioned difficulties by constructing complex line bundles over simplicial complexes representing M based on a formal approach. Following the classical definition of a bundle we define an atlas and associated bundle transition functions in terms of a discrete one-form on the dual mesh or — in other words — in terms of a discrete flat connection using the terminology from [43]. We show that the resulting atlas is well-defined in case the one-form is closed. This is necessary to ensure the correctness of the computations building upon this atlas.

As an application we compute smooth well-behaving embeddings of two-dimensional homology generators for any considered homology class. Those are invariant under isometric transformations and robust to noise and discretization.

Outline: In Sections 3–5 we discuss essential mathematical preliminaries in the smooth and discrete settings. Sections 6–8 describe the core of our approach. Section 9 summarizes the algorithm used for obtaining the results discussed in Section 10.

3. Basics

An appropriate mathematical setting for our discussion is provided by differential geometry, see e.g. [44], starting with a given Riemannian manifold M, possibly with boundary. For shape processing this manifold is typically, but not necessarily, embedded in an Euclidean space and can be pictured as a curve, surface, or volume. A differentiable manifold is usually defined in terms of an atlas, being a collection of open sets U_i covering M, together with chart homeomorphisms $U_i \to \mathbb{R}^n$ that induce differentiable chart transitions. The metric tensor, denoted by g or g_{ij} in local coordinates, allows for measuring metric properties such as lengths, angles and volumes on M. This tensor can be assumed to be given a-priori or to be induced by the embedding.

Commonly, vector bundles are introduced to equip the manifold with additional structure, see e.g. [44,45]. Intuitively, a rank k vector bundle E over a manifold M is obtained by assigning to each point $p \in M$ a k-dimensional vector space E_p in a continuous way. The vector space E_p is called fiber over p. Vector bundles of rank one are called line bundles. The prototypical example of a vector bundle is the tangent bundle TM which is the collection of all tangent spaces of M. Its dual is the cotangent bundle T^*M. Applying k times the exterior product to T^*M one obtains the bundles $\wedge^k T^*M$. A section of a bundle E is a differentiable map $s : M \to E$ with the property $s(p) \in E_p$ for all p. The space of sections of E is denoted by $\Gamma(E)$. These constructions are quite natural and familiar as for example $\Gamma(TM)$ is the space of vector fields and $\Gamma(\wedge^k T^*M)$, usually denoted by $\Omega^k(M)$, is the space of differential k-forms. The space of complex-valued functions or differential zero-forms on M can also be considered as the space of sections of the trivial line bundle $M \times \mathbb{C}$.

The most important operation on differential forms is the exterior derivative $d : \Omega^k \to \Omega^{k+1}$. Forms in the kernel of d are called closed, those in the image of d are called exact. Since $d^2 = 0$, the exterior derivative gives rise to the de Rham cohomology groups $H^k_{dR}(M)$ as the quotient groups of closed forms modulo