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Reconstruction of 3D objects from 2D cross-sections with the 4-point subdivision scheme adapted to sets

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ABSTRACT

Reconstruction of 3D objects from 2D cross-sections is an intriguing problem with many potential applications. We approach this problem through a novel multi-resolution method based on iterative refinement of the sets representing the cross-sections. To that end, we introduce a new geometric weighted average of two sets, defined for positive weights (corresponding to interpolation) and when one weight is negative (corresponding to extrapolation). This new average can be used to interpolate between cross-sections of a 3D object in a piecewise way. To obtain a smoother reconstruction of the 3D object, we adapt to sets the 4-point interpolatory subdivision scheme using the new average with both positive and negative weights. The effectiveness of the new method is demonstrated by several examples.

1. Introduction

Methods for reconstruction of objects from cross-sections have many applications. An important area of application is medical imaging [3,6,18], where 3D images are reconstructed from 2D slices obtained by medical imaging devices such as computational tomography (CT). Among other areas of application are computer graphics and animation, where sequences of intermediate shapes are created between two or more 2D or 3D shapes [21].

The problem of reconstruction from parallel cross-sections has been studied extensively for the past four decades and there exists a significant body of literature on this topic. Some methods, known as *parametric*, attempt to solve this problem by finding correspondence between points on the boundaries of the given cross-sections. Intermediate cross-sections are then created by interpolating positions of the sequences of corresponding points [4,5,23].

Another approach is to represent the cross-sections by *implicit functions* and then to interpolate between the implicit functions. A particularly popular method, originally proposed in [15], latter modified in [12,18] and extended in [7,8,16], is based on the representation of a cross-section by its *signed-distance function*. In this method the cross-sections are treated as sets, or equivalently as binary images. First, for each cross-section its signed distance function is computed. Next the signed-distance functions are pointwise interpolated by some univariate interpolation method, usually by linear or cubic spline interpolation. Finally the resulting function is thresholded at zero level, to obtain the reconstructed object.

Few works attempt to solve the problem by subdivision of sets. The main theme of this approach is the adaptation to sets of real valued subdivision methods (*see e.g.* [10]), by first expressing weighted averages between several numbers by sequences of weighted averages between two numbers (*binary weighted averages*). Binary weighted averages of numbers are then replaced by binary weighted averages of sets. In [9], spline subdivision schemes are adapted to sets using the *metric average* of sets. In [25], the Chaikin subdivision scheme is adapted to sets, based on the *straight-skeleton average* as a binary weighted average.

This work is a combination of the last two approaches. We develop a new weighted average of two sets based on the signed-distance function. The new binary weighted average is defined for positive weights and also when one weight is negative, therefore it performs both interpolation and extrapolation. Then we adapt to sets the 4-point interpolatory subdivision scheme [11], using the new binary average. Our numerical simulations demonstrate the quality of the reconstruction.

Although in this work we focus on the reconstruction of 3D objects from 2D cross-sections, our ideas are immediately applicable in any finite dimension. In particular, our method can be used to interpolate between a sequence of 3D objects.

The structure of this work is as follows. In Section 2 we introduce the new binary weighted average of sets, and interpolate piecewise between each two consecutive cross-sections. In Section 3, we adapt to sets the 4-point subdivision scheme and use it for the reconstruction of objects from cross-sections.

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Section 4 presents applications of our method to various data sets. The complexity of the method is discussed in Section 5. In Section 6, we draw conclusions and propose directions for future research.

2. The new binary average of sets

2.1. The average of "simply different" sets

2D sets representing cross-sections of a 3D object can be of complicated topology, see Fig. 1. We approach the construction of the new average of such sets by reducing it to several simple problems. First we consider the new average in the simple case of two sets $A,B \subset R^2$, such that one set is contained in the other, $B \subset A$, and the *difference set*,

$$A \backslash B = \{p \in A : p \notin B\},$$

has only one *connected component*. We call two such sets *simply different*. For two simply different sets A, B, one can think of the difference set $A \backslash B$ as a "vector" connecting the two sets. Therefore, when constructing a weighted average of two simply different sets, it is natural to try to mimic the very familiar case of a weighted average of two points in $\mathbb{R}^2$. Given two points $p,q \in \mathbb{R}^2$ it is easy to observe the following properties of the weighted average $tp+(1-t)q$, $t \in \mathbb{R}$:

1. The weighted average passes through q and p at $t=0$ and 1, respectively.
2. All averages are points on the same line. One way to express this idea is

   $$d_E(p_1,p_3) = d_E(p_1,p_2) + d_E(p_2,p_3),$$

   where $p_i = t_i p + (1-t_i)q$, $i=1,2,3$, $t_1 < t_2 < t_3$ and d_E is the Euclidean distance.
3. The average moves along the line with a constant velocity vector $(p-q)$.

We also observe that for $t \in [0,1]$ the average interpolates between the given points. For $t<0$, the average extrapolates from p beyond q, and for $t>1$ the average extrapolates from q beyond p. We aim to construct an average of two simply different sets satisfying the above three properties and extending the idea of interpolation and extrapolation.

Our construction is based on the *symmetric difference metric*, which is a widely used metric to measure dissimilarity between sets [20,22]. The *symmetric difference* of two sets A and B is

$$A \Delta B = (A \backslash B) \cup (B \backslash A),$$

and the *symmetric difference metric*[1] of $A,B \subset \mathbb{R}^2$ is defined by

$$d_{\mathfrak{L}}(A,B) = Area(A \Delta B).$$

First we introduce what we call the *distance average* of two simply different sets which we later modify to get our new average with the desirable properties. The distance average is based on the method of interpolation of the *signed-distance functions* introduced in [15]. The *signed distance* from a point p to a set A is defined by

$$d_S(p,A) = \begin{cases} d_E(p, Boundary(A)), & p \in A, \\ -d_E(p, Boundary(A)), & p \notin A, \end{cases}$$

with d_E the Euclidean distance from a point to a set. We use the signed-distance function, since in the literature it is a basic model for implicit representation of sets. We define the distance average between two simply different sets as the set,

$$tA \tilde{\oplus} (1-t)B = \{p : f_{A,B,t}(p) \geq 0\}, \tag{1}$$

where $f_{A,B,t}(p) = t d_S(p,A) + (1-t) d_S(p,B)$. Note that $f_{A,B,t}$ is not the signed-distance function of $tA \tilde{\oplus} (1-t)B$.

It is easy to see that the distance average passes though the original sets at $t=0$ and 1. We observe another important property of the distance average, which is

$$C_1 \cap C_3 \subseteq C_2 \subseteq C_1 \cup C_3, \tag{2}$$

with $C_i = t_i A \tilde{\oplus} (1-t_i)B$, $i=1, 2, 3$ and $t_1 \leq t_2 \leq t_3$. Relation (2) can be validated as follows. Let $p \in C_1 \cap C_3$, it follows from (1), that $f_{A,B,t_1}(p) \geq 0$ and $f_{A,B,t_3}(p) \geq 0$. Consequently, since $f_{A,B,t}(p)$ is linear as a function of t, for any $t_2 \in [t_1,t_3]$, $f_{A,B,t_2}(p) \geq 0$. So $p \in C_2$ and thus $C_1 \cap C_3 \subseteq C_2$. From similar considerations for the complement set of $C_1 \cup C_3$, we get $C_2 \subseteq C_1 \cup C_3$.

We observe also that all averages are "points" on an abstract "line" of sets due to the distance relation,

$$d_{\mathfrak{L}}(C_1,C_3) = d_{\mathfrak{L}}(C_1,C_2) + d_{\mathfrak{L}}(C_2,C_3), \tag{3}$$

which follows from (2) and Theorem 2 in [19]. However the distance average of sets lacks the constant velocity property, as can be observed from the following example. Consider the sets A and B shown in Fig. 2 as clipped together. Note that the difference set $A \backslash B$ has only one connected component (the inner oval), therefore A, B are simply different. Let p be a point in the interior of $A \backslash B$. For such a point:

$$d_S(p,A) > 0, \quad d_S(p,B) < 0 \quad \text{and} \quad |d_S(p,A)| > |d_S(p,B)|.$$

Therefore for any $p \in A$,

$$\tfrac{1}{2} d_S(p,A) + \tfrac{1}{2} d_S(p,B) > 0,$$

and consequently $\frac{1}{2}A \tilde{\oplus} \frac{1}{2}B = A$ (see Fig. 2). Indeed, it is undesirable to get one of the original sets, as an equally weighted average of the two different sets, since such average does not reflect a continuous transition of shape between the two sets.

To overcome this difficulty we construct a new weighted average of two simply different sets as a reparametrization of the distance average, imposing the constant velocity property. For this purpose we define the function,

$$g_{A,B}(x) = \text{sign}(x) \frac{d_{\mathfrak{L}}(xA \tilde{\oplus} (1-x)B, B)}{d_{\mathfrak{L}}(A,B)}. \tag{4}$$

It follows from (4) and (3), that the function $g_{A,B}$ is non-decreasing and $g_{A,B}(0)=0$, $g_{A,B}(1)=1$. The function $g_{A,B}$ reveals us how the average $xA \tilde{\oplus} (1-x)B$ is located relative to the set B (which corresponds to $x=0$). Next we define,

$$g_{A,B}^{-1}(t) = \underset{x}{\text{argmin}}\{|g_{A,B}(x) - t|\}. \tag{5}$$

Observe that if t belongs to the *range* of $g_{A,B}$, then

$$g_{A,B}(g_{A,B}^{-1}(t)) = t. \tag{6}$$

Fig. 1. A cross-section of CT scan of human pelvis.

[1] Technically, the "symmetric difference metric" is a metric on sets that are equal to the closure of their interior [20].