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Technical Section

# Ternary butterfly subdivision

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ABSTRACT

This paper presents an interpolating ternary butterfly subdivision scheme for triangular meshes based on a 1–9 splitting operator. The regular rules are derived from a $C^2$ interpolating subdivision curve, and the irregular rules are established through the Fourier analysis of the regular case. By analyzing the eigenstructures and characteristic maps, we show that the subdivision surfaces generated by this scheme is $C^1$ continuous up to valence 100. In addition, the curvature of regular region is bounded. Finally we demonstrate the visual quality of our subdivision scheme with several examples.



## 1. Introduction

Subdivision surfaces are valued in geometric modeling applications for their flexibility. Since 1978 [1,5], subdivision has been an active research area. Advances in computer memory have made subdivision methods practical in the late 1990s and this has prompted a large amount of new research work [2,9,16,25]. The flexibility primarily comes from the fact that objects to be subdivided can be of arbitrary topology and thus can be represented in a form that makes them easy for designing, rendering and manipulating.

Simply speaking, a subdivision surface is defined as the limit of a sequence of meshes. Each mesh in the sequence is generated from its predecessor using a group of topological and geometric rules. Topological rules are used to produce a finer mesh from a coarse one while geometric rules are designed to compute the positions of vertices in the new mesh. These two groups of rules constitute a subdivision scheme.

Subdivision can be distinguished into two classes: interpolating schemes [7,28] and approximating schemes [1,5,17,22]. If the old vertices are changed during refinement, the subdivision algorithm is considered to be approximating, otherwise it is interpolating. Although approximating algorithms yield limit surfaces with higher continuity, interpolating algorithms enjoy some obvious advantages which the approximating ones do not have: interpolating schemes are more efficient for the applications requiring interpolating specified vertices. Furthermore it is easy to generate multi-resolution surfaces by using interpolating schemes [26].

For a long time, interpolating schemes failed to generate surfaces with higher continuity. Although a 6-point interpolating scheme for a curve with $C^2$ continuity has been proposed in [24], it is not practical to extend this scheme to a surface because too many vertices would be included in a mask. Hassan et al. reported an interpolating ternary subdivision scheme for curves which achieves $C^2$ continuity [8]. The most desirable property is that only four points are needed to generate a new vertex. Starting from this curve case, Dodgson et al. [4] and Li and Ma [13] designed interpolating schemes for triangular meshes and quadrilateral meshes, respectively, but Li et al.'s scheme goes further in constructing surfaces with extraordinary vertices. It is well known that rules of a subdivision scheme for irregular meshes are particularly important if the scheme is to be used in practical applications [14,28], but unfortunately, irregular rules are not investigated in Dodgson et al.'s work, making it difficult to refine a coarse mesh with irregular vertices, which is often encountered in practice.

In this paper, we first slightly modify Dodgson et al.'s scheme, and then extend the regular rules to meshes with arbitrary topology through Fourier analysis. Based on the eigenstructures and characteristic maps, we show the $C^1$ continuity of subdivision surfaces for both regular and irregular regions up to valence 100. Due to the new 1–9 splitting operator, the face number increases

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by power of 9 in each step of the proposed subdivision. For this reason, we also explore the ability of adaptive subdivision. Finally some examples are presented to show the visual quality of the proposed subdivision scheme.

## 2. Related work

In 2002, Hassan et al. introduced interpolating ternary subdivision curves [8] shown in Fig. 1. The advantage of this new scheme is that it yields $C^2$ continuous limit curves. This subdivision scheme inserts two E-vertices into each edge of the given control polygon, respectively, at $\frac{1}{3}$ and $\frac{2}{3}$ parametric positions. A newly inserted vertex of the interpolating ternary subdivision scheme is computed as follows:

$$q_1 = a_0 p_{i-1}^k + a_1 p_i^k + a_2 p_{i+1}^k + a_3 p_{i+2}^k, \tag{1}$$

where

$$\begin{cases} a_0 = -\dfrac{1}{18} - \dfrac{1}{6}\mu, \\ a_1 = \dfrac{13}{18} + \dfrac{1}{2}\mu, \\ a_2 = \dfrac{7}{18} - \dfrac{1}{2}\mu, \\ a_3 = -\dfrac{1}{18} + \dfrac{1}{6}\mu, \end{cases} \tag{2}$$

with free parameter $\mu$. The mask for $q_2$ is symmetric to that of $q_1$. When $\frac{1}{9} \geq \mu \geq \frac{1}{15}$, the scheme generates $C^2$ limit curves [6].

Motivated by this subdivision curve, two subdivision schemes for surfaces have been proposed [4,13] consequently. Both of their ideas are to generalize the curve setting to surface configurations such that the regular rules can be derived by solving a system of equations. Although the irregular rules of [4] have been first investigated in [15], their result does not guarantee that the limit surfaces near extraordinary vertices are $C^1$ continuous. Furthermore, the eigenvalue analysis of irregular subdivision matrices in [15] takes only the 1-neighborhood of an extraordinary vertex into consideration. But according to the rules outlined in that paper, the smallest similar stencil should be 2-neighborhood, so that the eigenvalues provided in that paper cannot fully demonstrate the property of limit surfaces.

## 3. Regular subdivision masks

Our ternary subdivision introduces two types of new vertices: E-vertices, parametrically on the mesh edges, and F-vertices, parametrically at the face center. The regular rules of our subdivision scheme are similar to [4] except for a small modification. The regular masks for face vertex (F-vertex) and edge vertex (E-vertex) are presented in Figs. 2 and 3, respectively.

As mentioned above, the rules of Dodgson et al.'s scheme are derived from the ternary subdivision curve. The underlying surface masks reduce to the curve masks when the given control mesh collapses to a polyline along one of the three directions of the triangular mesh, so the weights in the masks must satisfy the

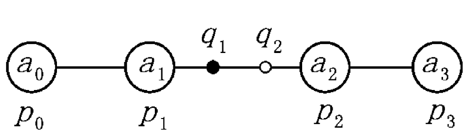


**Fig. 1.** Interpolating ternary subdivision curve: mask for newly inserted vertex $q_1$.

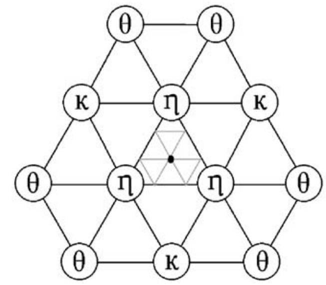


**Fig. 2.** Regular subdivision mask for a F-vertex $Q^F$ (the black dot).

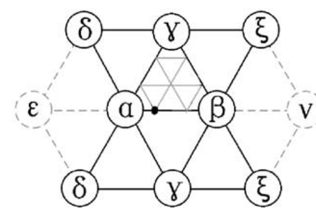


**Fig. 3.** Regular subdivision mask for an E-vertex $Q^E$ (the black dot), in which we suggest $\varepsilon$ and $\nu$ be zero.

following constraints:

$$\begin{cases} \kappa + 2\theta = a_0, \\ 2\eta + 2\theta = a_1, \\ \eta + 2\kappa = a_2, \\ 2\theta = a_3, \end{cases} \quad \begin{cases} \delta + \varepsilon = a_0, \\ \gamma + \alpha + \delta = a_1, \\ \xi + \beta + \gamma = a_2, \\ \nu + \xi = a_3. \end{cases} \tag{3}$$

By applying the conditions in Eq. (3), the mask with free parameters $\mu$, $\nu$, $\varepsilon$ can be obtained:

$$\begin{cases} \theta = \dfrac{1}{36}(-1 + 3\mu), \\ \kappa = \dfrac{1}{36}(-12\mu), \\ \eta = \dfrac{1}{36}(14 + 6\mu), \end{cases} \quad \begin{cases} \xi = \dfrac{1}{36}(-2 + 6\mu) - \nu, \\ \delta = \dfrac{1}{36}(-2 - 6\mu) - \varepsilon, \\ \gamma = \dfrac{1}{36}(4) + \varepsilon + \nu, \\ \beta = \dfrac{1}{36}(12 - 24\mu) - \varepsilon, \\ \alpha = \dfrac{1}{36}(24 + 24\mu) - \nu. \end{cases} \tag{4}$$

In order to simplify the analysis, we suggest that two weights, $\varepsilon$ and $\nu$, be zero. Then the shape of the E-mask is reduced to Dyn et al.'s butterfly scheme [7]. By analyzing the subdivision matrix with $\varepsilon = \nu = 0$, we can compute the eigenvalues by Mathematica 5.0 or other mathematic tools:

$$1, \frac{1}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, -\mu, \frac{1-5\mu}{6}, \frac{1-5\mu}{6}, \frac{1-5\mu}{6}, \frac{1-3\mu}{9}, \frac{1-3\mu}{12}, \frac{1-3\mu}{12}, \frac{1-9\mu}{18}, \frac{1-9\mu}{18}, \frac{1-9\mu}{18}, \frac{1-3\mu}{36}, \frac{1-3\mu}{36}, 0.$$

If $\mu$ is set to be $\frac{1}{11}$, the seventh eigenvalue is minimized and the 7th, 8th, 9th and 10th eigenvalues are then equal to $\frac{1}{11}$. The following eigenvalues are all less than $\frac{1}{11}$. For our proposed regular masks, we let $\mu$ be $\frac{1}{11}$ in this paper.

## 4. Masks near extraordinary vertices

### 4.1. Decomposition of regular masks

In order to simplify the derivation of irregular masks, we investigate the decomposition of the regular masks first. Figs. 4 and 5 show the process of decompositions. Three 1-neighborhood