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Continuous and discrete Mexican hat wavelet transforms on manifolds

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ABSTRACT

This paper systematically studies the well-known Mexican hat wavelet (MHW) on manifold geometry, including its derivation, properties, transforms, and applications. The MHW is rigorously derived from the heat kernel by taking the negative first-order derivative with respect to time. As a solution to the heat equation, it has a clear initial condition: the Laplace–Beltrami operator. Following a popular methodology in mathematics, we analyze the MHW and its transforms from a Fourier perspective. By formulating Fourier transforms of bivariate kernels and convolutions, we obtain its explicit expression in the Fourier domain, which is a scaled differential operator continuously dilated via heat diffusion. The MHW is localized in both space and frequency, which enables space-frequency analysis of input functions. We defined its continuous and discrete transforms as convolutions of bivariate kernels, and propose a fast method to compute convolutions by Fourier transform. To broaden its application scope, we apply the MHW to graphics problems of feature detection and geometry processing.

1. Introduction

One long-lasting task in geometry processing is to develop functional analysis tools on curved surfaces. Without Euclidean metric, it is extremely challenging to explicitly define functions on manifolds. Many existing methods are hinged upon differential geometry, where surface parameterization is frequently unavoidable. In this work, we study and develop functional analysis tools in frequency domain via spectral decomposition. Functions that have no closed-form expression on manifolds may have explicit formulations in frequency domain. The fundamental goal of this paper is to articulate this spectral approach with mathematical rigor, by studying the Mexican hat wavelet (MHW) and its transforms on manifolds.

Wavelet transforms are important tools for functional analysis and processing. One way to construct discrete wavelets on surfaces is via subdivision. As a regular domain with refining schemes, subdivision is convenient for subsampling and filter banks, which iteratively refine mesh geometry and functions. Subdivision wavelets heavily rely on subdivision connectivity of the mesh, which limits the application scope to data compression and level-of-detail rendering. The regularly-refined hierarchy is, however, computationally expensive and perhaps hard to build. Consequently, it gives rise to a strong demand in flexibly adapted wavelet tools without building the subdivision explicitly, which can be used for fast space-frequency analysis. For data compression, orthogonality is a crucial property of wavelets, while for space-frequency analysis, localization in both space and frequency is much more desirable. This requires wavelets are localized in space and frequency. It also implies the significance of analyzing functions in frequency domain.

In this paper, we advocate the well-known MHW on manifold geometry that is rigorously derived from heat diffusion. Analogous to the Euclidean MHW, the manifold MHW is defined as the negative first-order derivative of the heat kernel with respect to time. As a solution to the heat diffusion partial differential equation (PDE), it takes

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the Laplace–Beltrami operator as the initial condition. By defining Fourier transforms of bivariate kernels and convolutions, we further reveal that in the Fourier domain, the MHW is a product of the Laplace–Beltrami operator and the heat kernel. It is, therefore, a scaled differential operator continuously dilated through heat diffusion. It has Gaussian decays in both space and frequency, which implies it can extract information in a space-frequency window. We discuss some important properties of the MHW, such as admissibility, convergence, informativeness, and stableness. Moreover, we define its continuous and discrete transforms as convolutions of bivariate kernels. Similar with the case in signal processing, we propose a method to compute convolutions by Fourier transform, which significantly improves the computational time of wavelet transforms, without reducing their accuracy. Applications in feature detection and spectral geometry processing will immediately follow suit after we reveal the MHW's theoretic insights and document its most important properties. As an analog on manifold geometry, it is poised to excite more applications. While the central theme of this paper is studying the MHW and its transforms on manifolds, other contributions can be summarized as follows:

- We study Fourier transforms of bivariate kernels and convolutions on manifolds, with the purpose for function design in the Fourier domain.
- We approach the MHW transforms via Fourier decomposition. We show this Fourier method significantly reduces the complexity while preserving the accuracy.
- Based on the MHW theory, we formulate inverse transforms of continuous and discrete MHWs, which are concise and fast to compute.
- We devise immediate applications of the MHW in space-frequency analysis, including feature detection and spectral geometry processing.
- The proposed mechanism can be extended to other self-adjoint (differential) operators for functional analysis on manifolds.

2. Related work

This section briefly reviews previous work of Fourier transforms and wavelets adapted to manifold geometry.

2.1. Adapted Fourier transforms

Local areas of curved surfaces are homogeneous to 2D planar patches, where the Euclidean Fourier transform can be applied for spectral processing [20]. In terms of adapting the Fourier transform on manifolds, basis functions are critical for orthogonally decomposing the space to a series of shape spectra. In [2,13], eigenfunctions of the symmetric Laplacian of the connectivity graph are adopted as a Fourier basis, which is derived from the mesh topology but not the geometry. Analogous to the Fourier basis in Euclidean metric, manifolds have similar orthonormal basis formed by eigenfunctions of the Laplace–Beltrami operator [15]. Accordingly, Vallet and Lévy [27] defined the manifold harmonic transform (MHT) that is a fully adapted manifold Fourier transform, expanded on manifold harmonics (i.e., Laplace–Beltrami eigenfunctions). For applications, Rong et al. [22] employed this spectral decomposition to perform mesh editing on the base domain with low frequencies and reconstruct details with high frequencies. The Fourier basis, consisting of functions repeatedly oscillating over the entire domain, does not have localization in space. Therefore, adapted Fourier transforms only allow global operations of input functions.

2.2. Adapted wavelets

Defining wavelets on manifolds is never an easy task. One construction on meshed surfaces is achieved via explicit subdivision, which relies on the subdivision connectivity of the mesh. In [24], the lifting scheme is introduced for constructing subdivision wavelets on sphere. Lounsbery et al. [16] studied multiresolution analysis of wavelets constructed on surfaces of arbitrary topological type. In [3], B-spline wavelets are combined with the lifting scheme for biorthogonal wavelet construction. To avoid remeshing, Valette and Prost [26] extended the subdivision wavelet for triangular meshes using irregular subdivision scheme that can be directly computed on irregular meshes. On spherical domains, Haar wavelets [19] are constructed over nested triangular grids generated by subdivision. Recently, the spherical Haar wavelet basis was improved to the SOHO wavelet basis [14] that is both orthogonal and symmetric. In subdivision wavelets, the dilation of scaling functions strictly follows the subdivision scheme, which depends on the meshing. The subdivision wavelets have been frequently used for geometry compression and level-of-detail data visualization. It requires constructing the subdivision hierarchy before defining wavelets, which may limit its application scope. The regularly-refined hierarchy is computationally expensive and perhaps even harder to build.

Other than subdivision, a bottom-up construction of discrete diffusion wavelets [7] has been proposed on graphs and manifolds. They use a diffusion operator and its powers to build the nested subspaces, where scaling functions and wavelets are obtained by orthogonalization and rank-revealing compression. However, the constructed scaling and wavelet functions are not localized. In [18], the biorthogonal diffusion wavelets are introduced, relieving the excessively-strict orthogonality property of scaling functions. Rustamov [23] studied the relation between mesh editing and diffusion wavelets by introducing the generalized linear editing. The diffusion wavelets, iteratively constructed by matrix powers, are inconvenient for low-frequency processing.

In recent research results, mathematicians studied generating wavelets through the use of spectral theory. Hammond et al. [10] addressed graph wavelets through spectral graph theory. The graph wavelets are generated by a wavelet operator expanded on eigenfunctions of the graph Laplacian. In [1], Antoine et al. also studied continuous wavelet transforms on graphs, constructed by a generator in spectral domain. As an example, they introduced the Mexican hat wavelet formulated by the generator $u^2e^{-u^2}$ that is the Fourier transform of the Euclidean MHW. A similar result on compact manifolds was given