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Characterizations of simple points, simple edges and simple cliques of digital spaces: One method of topology-preserving transformations of digital spaces by deleting simple points and edges

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ABSTRACT

The notion of a simple point plays an important role in topology-preserving thinning, skeletonization and simplification of digital images. This paper presents new dimension-independent characterizations of simple points, simple edges and simple cliques based on the notion of a digital contractible space and contractible transformations of digital spaces. We show that a given digital space can be transformed to a normal digital space by the removal of simple points, edges and cliques while preserving topology. We describe a topology-preserving thinning algorithm, which transforms a given digital image to a normal one.

1. Introduction

Methods of digital topology are widely used in various image processing operations including topology-preserving thinning, skeletonization, simplification, border and surface tracing and region filling and growing.

Usually, transformations of digital objects preserve topological properties while changing the geometry of objects. One of the ways to do this in small dimensions is to use simple points: loosely speaking, a point of a digital object is called simple if it can be deleted from this object without altering topology. The detection of simple points is extremely important in image thinning, where a digital image of an object gets reduced to its skeleton with the same topological features.

The notion of a simple point was introduced by Rosenfeld [20]. Since then due to its importance, characterizations of simple points in two, three, and four dimensions and algorithms for their detection have been studied in the framework of digital topology by many researchers [1,3–5,8,15–19,21].

Local characterizations of simple points in three dimensions and efficient detection algorithms are particularly essential in such areas as medical image processing [2,9,10,22], where the shape correctness is required on the one hand and the image acquisition process is sensitive to the errors produced by the image noise, geometric distortions in the images, subject motion, etc., on the other hand.

Kong et al. [15] have studied properties of simple tiles and strongly normal collections of tiles in N dimensions (the extension of some of previous results for 2 and 3 dimensions) and have showed, in particular, that, in any strongly normal set of tiles, simpleness of a tile is equivalent to contractibility of its shared subset, and deletion of a simple tile preserves the homotopy type of the union of all the tiles.

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In paper [18], elements of a digital picture are associated with unitary cubes instead of points and techniques from cubical homology are applied to the cubical complex. Deleting a simple point means the removal of a unitary cube without changing the topology of the cubical complex. This approach allows characterizing a simple point in terms of the homology groups of the neighborhood of a point for arbitrary, finite dimensions.

Couprie and Bertrand [5] have introduced new characterizations of simple points in 2, 3 and 4 dimensions in the framework of cubical complexes and have shown that they lead to efficient algorithms for detecting such points.

A set of points is simple if the removal of this set from the object in which it lies does not change the homotopy type of the object. In paper [19], properties of the non-trivial simple sets composed of exactly two points have been studied.

In the present paper, we generalize and extend the notion of a simple point to the level, where it can be used in an arbitrary dimension, it does not assume the use of discrete grids Z^n and it can be applied to a complete set of points. We present the notion of a simple edge and a simple m-clique of a digital space based on digital contractible spaces and contractible transformations of digital spaces. We define a normal digital space: A digital space M is normal if it has no simple cliques. We prove that a digital space M is normal iff the nearest neighborhood of every point of M is a normal space. The advantage of using normal digital spaces is that it makes sense to study topological features not just of a normal space itself but, as well, of a local neighborhood of any point of the space.

We prove that a digital space can be transformed to a normal digital space by sequential removing simple points and edges and replacing simple cliques without changing topology of a space. We propose two topology-preserving thinning algorithms: for skeletonization and for normalization. The first one transforms a digital image into its skeleton; the second one transforms a digital image into a normal digital space.

2. Geometric background for simple adjacencies

The following surprising fact is observed in computer experiments modeling a motion and deformation of continuous surfaces and objects in three-dimensional Euclidean space [12]. Suppose that S_1 is a surface in Euclidean space E^3. Divide E^3 into a set of unit cubes with integer vertex coordinates and pick out the family M_1 of unit cubes intersecting S_1. Then construct the digital space D_1 corresponding to M_1 in the following way. D_1 corresponds to M_1 if there exists one–one onto correspondence between unit cubes in M_1 and points in D_1 that retains the adjacency relation between elements of M_1 and D_1. Remark that in graph theory, D_1 is called the intersection graph of M_1 [11]. Then slightly move and deform the surface S_1 into S_2. Reasoning as in the previous case, we construct D_2. After a series of motions and deformations, we obtain a sequence of digital spaces $D = \{D_1, D_2, \ldots, D_n\}$. It was revealed that any space in D could be turned into any other space in D by a sequence of transformations called contractible. The same tendency was observed when the scale of cubes was changed. At the same time, if families $D = \{D_1, D_2, \ldots, D_n\}$ and $C = \{C_1, C_2, \ldots, C_m\}$ represent, for example, a two-dimensional sphere and a three-dimensional ball, then no D_k can be converted into any C_i by contractible transformations. It was also observed that the Euler characteristic and the homology groups of all digital spaces in D coincided with the Euler characteristic and homology groups of the continuous counterparts. In the context of these experiments, one may assume that digital spaces contain topological and geometrical characteristics of continuous surfaces, and, in some way, transformations digitally model a homeomorphism of continuous spaces.

To illustrate our approach to the notion of a simple point, simple edge and a simple clique, let us consider the example depicted in Figs. 1 and 2.

The cubical family A (Fig. 1) consists of 44 unit cubes and is homotopic to a two-dimensional (continuous) sphere, Obviously, the removal of cubes a, b and all other cubes similar to a and b does not change the topology of A. As a consequence, cubical families A and D have the same topology (Fig. 1). The digital image $I(D)$ of D will be the intersection graph of the family of cubes of D. D has no simple cubes and $I(D)$ has no simple points. The digital image $I(E)$ of the collection $E \subseteq D$ of cubes adjacent to the cube 1 (Fig. 2) is a non-contractible space, containing simple points 4, 6 and 11. The purpose of this article is to find transformations, which would remove such simple points from $I(E)$ without altering the topology of $I(D)$. Obviously, cubes 4, 6 or 11 cannot be removed from D because the removal will change the topology of D.

The removal of the simple point 4 from $I(E)$ means the removal of the edge $(1, 4)$ in the digital image $I(D)$. Note that the collection F of cubes adjacent to both cubes 1 and 4 consists of two cubes 2 and 3 and the digital image $I(F)$ is a contractible space. According to properties of contractible transformation, the edge $(1, 4)$ can be deleted from $I(D)$ [14]. Geometrically, this means that cubes 1 and 2 in D should be made non-adjacent while preserving the topology of D. One way to do this is to resize necessary cubes, for example, 1–4 into a collection of cuboids with required properties (Fig. 2). Finally, the cuboid set G has the same topology as cubical sets A and D, the digital images of $I(A)$, $I(D)$ and $I(G)$ are homotopic to each other and G (and $I(G)$) has no simple cuboids (points) and adjacencies. Thus, we reconstruct the cubical set A into the cuboid set G with the same topology as of A and without simple elements and adjacencies. Notice that there are many ways to realize the digital space $I(G)$ geometrically and the cuboid family is only one of them. The removal of a simple point (edge) must be done sequentially in a single step because this can change the simplicity of a nearest point or edge.

3. Preliminaries

To make this paper self-contained, we summarize the necessary information from previous papers.

A digital space G is a simple undirected *graph* $G = (V, W)$, where $V = \{v_1, v_2, \ldots, v_n, \ldots\}$ is a finite or countable set of points, and $W = \{(v_p v_q), \ldots\} \subseteq V \times V$ is a set of edges