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Extraction of cylinders and cones from minimal point sets☆

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ABSTRACT

We propose new algebraic methods for extracting cylinders and cones from minimal point sets, including oriented points. More precisely, we are interested in computing efficiently cylinders through a set of three points, one of them being oriented, or through a set of five simple points. We are also interested in computing efficiently cones through a set of two oriented points, through a set of four points, one of them being oriented, or through a set of six points. For these different interpolation problems, we give optimal bounds on the number of solutions. Moreover, we describe algebraic methods targeted to solve these problems efficiently.

1. Introduction

Extracting geometric primitives from 3D point clouds is an important problem in reverse engineering. These 3D point clouds are typically obtained from accurate 3D scanners and there exist several methods for extracting 3D geometric primitives [2]. An important category among these methods is based on the RANSAC approach [2,8,11]. For such methods, the primitives are extracted directly from the input point cloud. The basic idea is to extract a particular elementary type of shape, such as planes, spheres, cylinders, cones or tori, from the smallest possible set of points and then to judge if this extracted primitive is relevant to the full point cloud. Therefore, for this category of methods it is very important to compute a particular type of shape through the smallest possible number of points, including normals if available. If extracting planes and spheres is easy, the cases of cylinders and cones are more involved. In this paper we provide new methods for extracting these geometric primitives from the smallest possible number of points, counting multiplicities of oriented points (i.e. points given with their normal vector). These methods are intended to serve the larger goal of improving speed and numerical accuracy in data extraction from graphical information. As far as we know, and surprisingly, the above-mentioned problems have not appeared in the existing literature with the exception of [6,9]. Instead, the classical approaches to these interpolation problems usually extract, actually we should say estimate, these geometric primitives from an overdetermined number of points, counting multiplicities (e.g. [10]).

An *oriented point* is a couple of a point and a nonzero vector. A surface is said to interpolate an oriented point if the point belongs to the surface and its associated vector is collinear to the normal of the surface at this point, we do not assume that the orientation of the normal of the point is the same as the orientation of the surface since often in the data sets normals are unoriented. Moreover, it is important to deal with inhomogeneous data, that is to say some points are oriented but not all, in order to take into account the estimated accuracy of oriented point clouds that are generated by means of normal estimation algorithms. Data made of points and oriented points will be called a *mixed set of points*.

We emphasize that interpolating at a point imposes a single algebraic condition on a given shape whereas interpolating at an oriented 3D point imposes three algebraic conditions. Typically, a 3D plane is uniquely defined either by three distinct points or by one oriented point. A sphere is uniquely defined either by four points or by one oriented point and an additional point. In these two cases, it turns out that there is a unique shape that interpolates a mixed set of points corresponding to the number of parameters of this shape (a plane is determined by three parameters and the sphere is determined by four parameters). In this paper, we will treat interpolation of two other basic shapes, namely cylinders and cones for which the situation is more involved.

Our approach is inspired by effective methods in algebraic geometry. We consider two families of unknowns. The first one corresponds to the parameters needed to describe all features of the

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targeted surface (e.g. the radius and axis of a circular cylinder) and hence its equation. The second family consists of auxiliary unknowns (e.g. such as a special point on that axis) which permit us to describe a collection of geometric constructions. These constructions are designed to establish a complete link between the input and the first family of unknowns. Then, we translate algebraically the collection of constraints attached to theses geometric constructions into a system of polynomial equations that we further analyze and simplify, discarding spurious solutions if necessary. Since the input and output are (and should be) real approximate data, we designed efficient algorithms to compute very accurate real solutions of these systems of equations. Indeed, in all the considered cases, we were able to express the results as the solutions of (generalized) eigenvalue problems together with close formulas. These expressions allow us to rely on classical matrix computation software and achieve accuracy and efficiency. Prototypes of our algorithms are implemented in the computer algebra system MAPLE, and we provide some statistics and timings (which are quite satisfactory).

2. Interpolation of cylinders

A cylinder (more precisely a right circular cylinder) is defined as the set of points in the three-dimensional affine space $\mathbb{R}^3$ located at a fixed distance (called the *radius* of the cylinder) of a given straight line (called the *axis* of the cylinder). It is hence defined by means of five parameters : four parameters describe a line in $\mathbb{R}^3$ and an additional parameter measures the radius.

A popular determination of a cylinder is done by interpolating two points with normals, which imposes six conditions (instead of five). So, a priori no cylinder interpolates this data; therefore some approximations are necessary. In this section, we will give new methods to compute cylinders using just five independent conditions. There are two possible such minimal configurations, either an oriented point and two other distinct points, or five distinct points.

2.1. Cylinders through a mixed minimal point set

We seek the cylinders that interpolate a given mixed minimal set of points $\mathcal{P}$. Since a cylinder is given by 5 parameters, $\mathcal{P}$ is assumed to be composed of an oriented point p_1 with its normal vector n_1 and two other distinct points p_2, p_3 in $\mathbb{R}^3$.

First, by a linear change of coordinates, one can assume that $p_1 = (0, 0, 0)$ and $n_1 = (0, 0, 1)$ and we set $p_2 = (x_2, y_2, z_2)$ and $p_3 = (x_3, y_3, z_3)$. Then, the axis of a cylinder interpolating $\mathcal{P}$ must be orthogonal to the z-axis and must intersect it. It follows that a normal plane Π contains the z-axis, hence is given by an equation of the form $lx + my = 0$ where $t := (l, m, 0)$ is the corresponding direction of the axis. Observe that these directions are in correspondence with a projective line $\mathbb{P}^1$. For simplicity, we set $\rho := \sqrt{l^2 + m^2} = \|t\| > 0$.

Now, we compute the orthogonal projections q_1 and q_2 of p_2 and p_3 onto the plane Π. Π contains the point p_1 and is generated by the two orthogonal vectors n_1 and $v = n_1 \wedge t = (-m, l, 0)$. The matrix

$$M = \begin{pmatrix} \frac{l}{\rho} & \frac{m}{\rho} & 0 \\ -\frac{m}{\rho} & \frac{l}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines the change of coordinates from the current coordinate system (x, y, z) to a new coordinate system (x', y', z'), with the same origin p_1, defined by the three vectors t/ρ, v/ρ, n_1, where Π has equation $x' = 0$. It follows that the coordinates (x'_i, y'_i, z'_i) of p_i, $i = 2, 3$, in this new coordinate system are given by

$$(x'_i, y'_i, z'_i) = \left(x_i \frac{l}{\rho} + y_i \frac{m}{\rho}, -x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right).$$

Therefore, the coordinates of the orthogonal projections q_i, $i = 2, 3$ are given by

$$q_i = \left(-x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right) \in \Pi$$

in the basis v/ρ, n_1.

The existence of a cylinder interpolating $\mathcal{P}$ is equivalent to the fact that the points p_1, q_2 and q_3 all belong to a circle whose center c is located on the z-axis, say $c = (0, 0, r)$. Such a circle has an equation of the form $y'^2 + (z' - r)^2 = r^2$, or equivalently $y'^2 + z'^2 - 2rz' = 0$. Therefore, this cocyclicity condition can be written as

$$\begin{aligned} 0 &= \begin{vmatrix} y_2'^2 + z_2^2 & z_2 \\ y_3'^2 + z_3^2 & z_3 \end{vmatrix} \\ &= \frac{1}{\rho^2} \begin{vmatrix} (-x_2 m + y_2 l)^2 + (l^2 + m^2) z_2^2 & z_2 \\ (-x_3 m + y_3 l)^2 + (l^2 + m^2) z_3^2 & z_3 \end{vmatrix}. \end{aligned}$$

Since $\rho > 0$, the expansion of this latter determinant allows us to rewrite this condition as a degree 2 homogeneous equation $al^2 + blm + cm^2$ where the coefficients a, b, c are given by the following closed formulas

$$a := \begin{vmatrix} y_2^2 + z_2^2 & z_2 \\ y_3^2 + z_3^2 & z_3 \end{vmatrix}, \quad b := -2 \begin{vmatrix} x_2 y_2 & z_2 \\ x_3 y_3 & z_3 \end{vmatrix},$$

$$c := \begin{vmatrix} x_2^2 + z_2^2 & z_2 \\ x_3^2 + z_3^2 & z_3 \end{vmatrix}.$$

Unless $a = b = c = 0$, this equation has two roots, counting multiplicities, in the field of complex numbers. If a real solution is found, that is to say the direction of a real cylinder interpolating $\mathcal{P}$ (observe that one can impose $\rho = 1$ since the condition is homogeneous in l, m), then the remaining parameter r is uniquely determined by one of the formulas

$$2z_2 r = y_2'^2 + z_2^2 = \frac{1}{\rho^2}(-x_2 m + y_2 l)^2 + z_2^2, \tag{1}$$

$$2z_3 r = y_3'^2 + z_3^2 = \frac{1}{\rho^2}(-x_3 m + y_3 l)^2 + z_3^2, \tag{2}$$

depending on whether $z_2 \neq 0$ or $z_3 \neq 0$. We notice that if $z_2 = z_3 = 0$ then $a = b = c = 0$.

Theorem 1. *Given a mixed set of points $\mathcal{P}$ composed of an oriented point p_1, n_1 and two other points p_2, p_3 such that*

(i) *p_1, p_2, p_3 are all distinct,*
(ii) *p_1, p_2, p_3 do not belong to a common plane that is normal to n_1,*
(iii) *p_2 and p_3 are not symmetric with respect to the line through p_1 and generated by n_1,*

then there exist at most 2 real cylinders interpolating $\mathcal{P}$. Otherwise, there exists a cylinder (possibly "flat", i.e. with infinite radius) interpolating $\mathcal{P}$ in any direction that is normal to n_1.

Proof. Following the above discussion, this theorem will be proved if we show that $a = b = c = 0$ if and only if at least one of the three conditions (i), (ii), (iii) holds. It is not hard to check that if one of the three latter conditions holds then $a = b = c = 0$. To prove the converse, we observe that by a linear change of coordinate, we can assume that $x_2 = 0$ in addition to the fact that $p_1 = (0, 0, 0)$ and