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Recognizing projections of algebraic curves

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ABSTRACT

Given two irreducible, algebraic space curves $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_2$ is contained in some plane Π, we provide algorithms to check whether or not there exist perspective or parallel projections mapping $\mathcal{C}_1$ onto $\mathcal{C}_2$, i.e. to recognize $\mathcal{C}_2$ as the projection of $\mathcal{C}_1$. In the affirmative case, the algorithms provide the eye point(s) of the perspective transformation(s), or the direction(s) of the parallel projection(s). Although the problem is mainly discussed for rational curves, an algorithm for implicit curves is also given. The algorithms presented are mostly symbolic; nevertheless, we include an approximate algorithm for rational curves too.

1. Introduction

In this paper, we address the following geometric problem: *given* an irreducible, algebraic space curve $\mathcal{C}_2$, lying on a plane Π, and another irreducible, algebraic space curve $\mathcal{C}_1$, not necessarily planar, *check* whether or not there exist perspective or parallel projections mapping $\mathcal{C}_1$ onto $\mathcal{C}_2$, and *find* them in the affirmative case.

Our problem can be translated into the context of Computer Vision. For this purpose, we recall [16] the simplest camera model, known as the *pinhole camera*. In this model, a camera is modeled as a pair $(\mathbf{a}, \mathbf{\Pi})$, where $\mathbf{a}$ is called the *eye point* of the camera, and $\mathbf{\Pi}$ is the *image plane*: then, given an object $\Omega \subset \mathbb{R}^3$, the photograph of Ω taken by the camera is the projection of Ω from the point $\mathbf{a}$ onto the plane $\mathbf{\Pi}$ (see Fig. 1). The eye point can be allowed to be at infinity, in which case we have a parallel projection from a certain direction.

Therefore, in this context our problem can be translated as whether or not $\mathcal{C}_2$ can be regarded as a *photograph* of $\mathcal{C}_1$, taken with a camera where the image plane is known (it is the plane Π containing $\mathcal{C}_2$), but where the eye point is unknown.

A more general problem is treated in [8]. In [8] the input is a pair of algebraic curves, $\mathcal{D}_1 \subset \mathbb{R}^3$ and $\mathcal{D}_2 \subset \mathbb{R}^2$, and the question is to check if there exists some camera where $\mathcal{D}_2$ is the photograph of $\mathcal{D}_1$: in other words, to find the positions of the eye point *and* the image plane, if any, such that $\mathcal{D}_2$ is the photograph of $\mathcal{D}_1$ taken from the camera eye point. This problem is known as the *object-image correspondence* problem, and amounts to recognizing images without any clue on the parameters of the camera used to take the photograph.

In [8] the problem is solved by deciding whether the curve $\mathcal{D}_2$ is equivalent to some curve in a family of planar curves, computed from $\mathcal{D}_1$, under an action of the projective or the affine group. In turn, this is done by using differential invariants. Computationally, the question boils down to solving a quantifier elimination problem with five variables, in the case of perspective projections, and with four variables in the case of parallel projections. Since the differential invariants used in [8] are high-order (5 and 6 for affine actions, 7 and 8 for projective actions), the elimination problem can be hard.

In our case we assume that the image plane is known, so our problem can be considered as a weak version of the problem in [8]. Because the problem is simpler, we can find a solution computationally simpler than that of [8], too, especially in the case of rational curves. In the rational case, we restrict to rational curves properly parametrized over $\mathbb{Q}$, and use a very different approach to that in [8]. We observe that any projection between $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponds to a rational function ψ between the parameter spaces; furthermore, ψ is shown to be a Möbius transformation in the case of non-degenerate projections between $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e. projections which are injective for almost all points of $\mathcal{C}_2$. We show that the rational functions ψ potentially corresponding to projections from eye points with rational coordinates, can be efficiently computed by means of standard bivariate factoring techniques over the rationals. In order to also find the projections from non-rational eye

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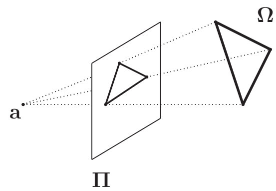

Fig. 1. Pinhole camera model.

points we need bivariate factoring over the reals, i.e. an absolute factorization. Furthermore, once ψ is computed, checking if it gives rise to some projection between $\mathcal{C}_1$ and $\mathcal{C}_2$ is easy.

As an alternative to computing an absolute factorization, also in the rational case, we provide another algorithm, both in symbolic and approximate versions, which takes advantage of the existence of ψ without actually computing it. In general, the symbolic version of this last algorithm requires to compute the primitive element of an algebraic extension $\mathbb{Q}(\alpha, \beta)$, which can be costly. However, the approximate version of the algorithm, where algebraic numbers are numerically approximated, is fast. In this last case, we do not check if $\mathcal{C}_2$ is the projection of $\mathcal{C}_1$, but if $\mathcal{C}_2$ is "approximately" the projection of $\mathcal{C}_1$. Furthermore, this approximate algorithm is well-suited for curves whose defining parametrizations are known only up to a certain precision, i.e. with floating point coefficients, which is closer to applications.

In the case of implicit curves, we also present a symbolic algorithm for solving the problem. In this case the algorithm uses Gröbner bases, and is only suitable for low degree curves. For higher degrees, the Gröbner elimination process can be costly.

2. Generalities on projective and parallel projections.

Throughout the paper, we consider two irreducible, algebraic space curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^3$, $\mathcal{C}_1 \neq \mathcal{C}_2$. We will suppose that $\mathcal{C}_2$ is contained in some plane Π; however, $\mathcal{C}_1$ is not necessarily planar.

We first address the case when $\mathcal{C}_1$ and $\mathcal{C}_2$ are rational, i.e. parametrized by rational maps

$$\boldsymbol{x}_j : \mathbb{R} \dashrightarrow \mathcal{C}_j \subset \mathbb{R}^3, \quad \boldsymbol{x}_j(t) = \big(x_j(t), y_j(t), z_j(t)\big), \quad j = 1, 2. \tag{1}$$

Here, we will exclude the case when $\mathcal{C}_1, \mathcal{C}_2$ are two planar curves contained in the *same* plane. Additionally, we will suppose that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ have coefficients in $\mathbb{Q}$. The components x_j, y_j, z_j of $\boldsymbol{x}_j$ are real, rational functions of t, therefore defined for all but a finite number of values of t. Nevertheless, at certain moments we will consider x_j, y_j, z_j as functions from $\mathbb{C}$ to $\mathbb{C}$. We will assume that the parametrizations in (1) are *proper*, i.e., birational or, equivalently, injective except for perhaps finitely many values of t. This can be assumed without loss of generality, since any rational curve can be properly reparametrized. For these claims and other results on properness, the interested reader can consult [24] for plane curves and [1, §3.1] for space curves.

Perspective and *parallel* projections onto a plane Π are well-known transformations in 3-space, illustrated in Fig 2. Perspective projections onto Π are projections from a point, called the *eye point*, while parallel projections are projections in the direction of a nonzero vector $\mathbf{v} \in \mathbb{R}^3$. Both types of projections can be

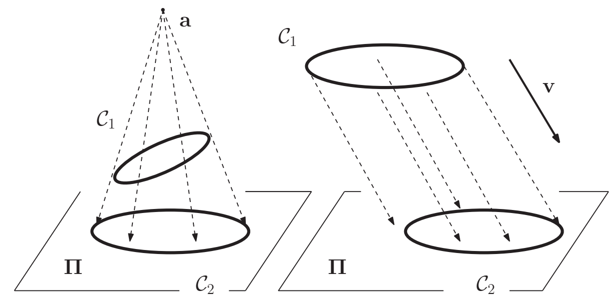

Fig. 2. Perspective projection (left), parallel projection (right).

unified when we move to a projective setting. In (complex) projective space $\mathbb{P}^3_{\mathbb{C}}$, the *projective closure* of $\mathcal{C}_j$ is the curve in $\mathbb{P}^3_{\mathbb{C}}$ whose affine part is $\mathcal{C}_j$. If $\mathcal{C}_j$ is rational, the projective closure can be parametrized as

$$\tilde{\boldsymbol{x}}(t) = \left[\tilde{x}_j(t) : \tilde{y}_j(t) : \tilde{z}_j(t) : \tilde{\omega}_j(t)\right],$$

where $x_j(t) = \frac{\tilde{x}_j(t)}{\tilde{\omega}_j(t)}$, $y_j(t) = \frac{\tilde{y}_j(t)}{\tilde{\omega}_j(t)}$, $z_j(t) = \frac{\tilde{z}_j(t)}{\tilde{\omega}_j(t)}$. For simplicity we will use the same notation for a curve and its projective closure; it will be clear from the context whether we are working with one or the other.

In projective space, parallel or perspective projections are treated in the same way: in the case of perspective projections the eye point is affine, and in the case of parallel projections, the eye point is at infinity. So both projections [13, §13] can be represented by a projective transformation

$$\begin{bmatrix} x' \\ y' \\ z' \\ \omega' \end{bmatrix} = \underbrace{\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}}_{P} \cdot \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix}.$$

If $\tilde{\mathbf{a}} := [\tilde{a}_1 : \tilde{a}_2 : \tilde{a}_3 : \tilde{a}_4]$ denotes the eye point of the projection, and the implicit equation of the projection plane Π is $Ax + By + Cz + D = 0$, an easy computation shows that the matrix P is

$$P = \begin{bmatrix} -\tilde{a}_2B - \tilde{a}_3C - \tilde{a}_4D & \tilde{a}_1B & \tilde{a}_1C & \tilde{a}_1D \\ \tilde{a}_2A & -\tilde{a}_1A - \tilde{a}_3C - \tilde{a}_4D & \tilde{a}_2C & \tilde{a}_2D \\ \tilde{a}_3A & \tilde{a}_3B & -\tilde{a}_1A - \tilde{a}_2B - \tilde{a}_4D & \tilde{a}_3D \\ \tilde{a}_4A & \tilde{a}_4B & \tilde{a}_4C & -\tilde{a}_1A - \tilde{a}_2B - \tilde{a}_3C \end{bmatrix}. \tag{2}$$

In the paper we will work with *real* projections and real curves $\mathcal{C}_1, \mathcal{C}_2$, so that the eye point, and therefore the matrix P, are real.

Given a projection $\mathcal{P}_{\tilde{\mathbf{a}}}$ onto a plane Π and a curve $\mathcal{C}$, the *projection of* $\mathcal{C}$, $\mathcal{P}_{\tilde{\mathbf{a}}}(\mathcal{C})$, is the image of $\mathcal{C}$ under $\mathcal{P}_{\tilde{\mathbf{a}}}$. Therefore, *our goal is to check if $\mathcal{C}_2$ is the projection of $\mathcal{C}_1$ from some eye point $\tilde{\mathbf{a}}$ onto the plane Π containing $\mathcal{C}_2$, i.e. if there exists $\tilde{\mathbf{a}}$ such that $\mathcal{C}_2 = \mathcal{P}_{\tilde{\mathbf{a}}}(\mathcal{C}_1)$, and find $\tilde{\mathbf{a}}$ in the affirmative case*. Notice that the solution is not necessarily unique. For instance, let $\mathcal{C}_1$ be $\{x^2 + y^2 = 1, z = 1\}$ and let $\mathcal{C}_2$ be $\{x^2 + y^2 = 2, z = 0\}$ (see Fig. 3). These curves are two circles of radii 1 and $\sqrt{2}$, located in the planes $z = 1$ and $z = 0$, with centers on the z-axis. In this case there are two different perspective projections transforming $\mathcal{C}_1$ into $\mathcal{C}_2$, one from the point $\left(0, 0, \frac{\sqrt{2}}{\sqrt{2}-1}\right)$, and another one from the point $\left(0, 0, \frac{\sqrt{2}}{\sqrt{2}+1}\right)$. This example also shows that two rational curves $\mathcal{C}_1$ and $\mathcal{C}_2$ parametrized over $\mathbb{Q}$ can however be related by a projection from a point with non-rational coordinates.

If $\mathcal{P}_{\tilde{\mathbf{a}}}|_{\mathcal{C}_1}$ is injective for almost all points of $\mathcal{C}_1$, so there are not two different branches of $\mathcal{C}_1$ whose projection onto $\mathcal{C}_2$ overlap, we will say that $\mathcal{P}_{\tilde{\mathbf{a}}}$ is *non-degenerate*; otherwise, we will say