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Survey on sparsity in geometric modeling and processing

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ABSTRACT

Techniques from sparse representation have been successfully applied in many areas like digital image processing, computer vision and pattern recognition in the past ten years. However, sparsity based methods in geometric processing is far from popular than its applications in these areas. The main reason is that geometric signal is a two-dimensional manifold and its discrete representations are always irregular, which is different from signals like audio and image. Therefore, existing techniques cannot be directly extended to handle geometric models. Fortunately, sparse models are beginning to see significant success in many classical geometric processing problems like mesh denoising, point cloud compression, etc. This review paper highlights a few representative examples of how the interaction between sparsity based methods and geometric processing can enrich both fields, and raises a number of open questions for future study.

1. Introduction

Sparsity based regularization and sparse signal representation [1,2] have proven to be an extremely powerful tool for processing signals like audio, image and video. Sparse techniques have become state-of-the-art tools in many fields like machine learning [3,4], signal processing [5,6], neuroscience [7,8] and statistics [9–11]. This success is mainly due to the fact that important classes of signals such as audio and images have naturally sparse representations with respect to fixed bases (i.e., Fourier, Wavelet), or concatenations of such bases. Moreover, efficient and provably effective algorithms based on convex optimization or greedy pursuit are available for computing such representations with high fidelity [12].

While these successes in classical signal processing applications are inspiring, in geometric processing we are dealing with two-dimensional manifold signals with irregular domain, which is totally different from audio, image and video. One might justifiably wonder whether sparsity based regularization and sparse representation can be useful at all for geometric processing tasks. The answer has been largely positive: in the past few years, variations and extensions of ℓ_1 minimization have been applied to many geometric processing tasks, including mesh denoising [13–15], surface reconstruction [16], point cloud consolidation [17–21], mesh segmentation [22–25], and point cloud registration [26]. In almost all of these applications, using sparsity as a prior leads to state-of-the-art results.

Before going any further, we would like to briefly analyze the difference between using sparse techniques in geometry and in traditional fields. Sparse signal techniques have been successfully applied on many aspects as acquiring, representing and compressing high-dimensional signals. This is because that signals like audio and images can be sparsely represented by fixed basis like Fourier, Wavelet and Discrete Cosine Transform (DCT). Another important property of these signals is that they have a natural domain on which functions can be defined. For instance the domain of an audio is time or frequency and the domain of an image is a regular planar grid.

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Geometric signals usually consist of geometric positions and sometimes connection relationships. The connection relationships are represented by a 3D graph on which we can take geometric positions as sampling for a certain three dimensional functions. We can take these relationships as the domain of geometric signals. However these domains usually cannot embed onto a planar region and they are irregular compared to previous domains. Sometimes we cannot handle geometry directly and we have to transform the geometry into feature space which might be some Euclidean space. With these irregular domains another important issue is that how to define the basis. Besides, famous sparse related regularization terms for image processing like TV model assume that image is piece-wise constant. Geometric signals on the other hand are at least continuous. Thus most regularization terms cannot be directly employed on geometric problems.

Above all, applying sparse techniques on geometric signals generally faces the problems of handling irregular domain, defining basis functions and the geometric specified regularization terms. Fortunately, many creative researchers have found a lot of effective methods tackling these problems and successfully used sparse techniques to solve geometric problems as mentioned above. And the experiment results sufficiently show the advantages of sparse techniques, such as robustness to noise, local controllability and feature preserving.

In the rest of this paper, we would like to first introduce traditional sparse models (Section 2) used in previous fields like machine learning, computer vision, etc. Then according to the different sparse models, we classify all the papers into three parts (Sections 3–5) where we illustrate how these techniques are successfully applied on geometric processing problems. By giving a survey about the usage and the effectiveness of sparse techniques, we would like to achieve the goal of inspiring the researchers who are interested to discover more applications. In the end, Section 6 gives a summary and possible future works.

2. Preliminaries

Before illustrating how sparsity is applied on geometry processing problems, we would like to introduce some notations and general sparsity models.

2.1. Notation

To make this survey self-contained, here we introduce some basic notations. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)^T$ be any vector in Euclidean space $\mathbb{R}^k$, $\|\mathbf{x}\|_p$ denotes the ℓ_p norm of $\mathbf{x}$ with $\|\mathbf{x}\|_p = (\sum_{i=1}^k |x_i|^p)^{1/p}$. And the ℓ_0 pseudo-norm of $\mathbf{x}$ is defined as $\|\mathbf{x}\|_0 = \#\{i|x_i \neq 0\} = \sum_{i=1}^k |x_i|^0$. $\mathbf{M} = (m_{ij})$ represents a matrix in $\mathbb{R}^{m\times n}$. Its frobenius norm is defined as $\|\mathbf{M}\|_F = (\sum_{i=1}^m \sum_{i=1}^n m_{ij}^2)^{1/2}$, and its nuclear norm is defined as $\|\mathbf{M}\|_* = \sum_i \sigma_i(\mathbf{M})$ where $\sigma_i(\mathbf{M})$ is the ith singular value of $\mathbf{M}$. Nuclear norm is the convex envelope of rank($\mathbf{M}$), which makes that $\|\mathbf{M}\|_*$ can be considered the relaxation of the rank of $\mathbf{M}$.

2.2. Sparse techniques

Generally, there are some basic assumptions in sparse techniques. For example, a signal can be represented by a sparse linear combination of dictionary elements, some special signals can be approximated by a low rank matrix. In the following, we will discuss these essential issues and general models raised in this field.

2.2.1. Sparsity in vector

A vector signal $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ can be approximated by a linear combination of dictionary elements $\{\mathbf{d}_i \in \mathbb{R}^n\}_{i=1}^m$, which can be formulated as

$$\mathbf{u} \approx \sum_{i=1}^{m} x_i \mathbf{d}_i, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ is the coefficient vector. If $\mathbf{x}$ is sparse, it means that signal $\mathbf{u}$ can be represented by a linear combination of few dictionary elements. However there are also different explanations about sparsity that the vector becomes sparse under a certain transformation. For instance, the gradient of natural image is always sparse, and total variation model catches this observation well. In the following, we will give three general models which are quite popular in signal processing.

Sparse coding. Sparse coding method is widely used in computer vision tasks like face recognition, image classification. It assumes that the input signal can be sparsely represented by a set of dictionary elements. The target of sparse coding is to pursuit the sparse coefficient vector $\mathbf{x}$. The formulation is

$$\min_{\mathbf{x}} \frac{\lambda}{2} \|\mathbf{u} - \mathbf{D}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_p, \tag{2}$$

where $0 \le p \le 1$. If $p = 0$, $\|\mathbf{x}\|_p$ is equivalent to the number of non-zero elements. However $\|\mathbf{x}\|_0$ is a nonconvex norm such that it is quite hard to obtain the optimal result and most methods use greedy strategy to get an approximation result [27,28]. On the other hand, Eq. (2) would be a convex problem if we set $p = 1$, and it is the well-known least absolute shrinkage and selection operator (LASSO) [29]. The relationship of the ℓ_1 relaxation and its origin sparse ℓ_0 model is an open problem and [30] proves that under certain conditions the results are equivalent. Recently researchers develop algorithms solving (2) when $0 < p < 1$ which also approximates the sparse solution. As shown in Fig. 1, the iso-level curve of $\|\mathbf{x}\|_p = 1$ concentrates toward axes with p decreasing, and thus model (2) returns more sparse result with smaller p value.

As shown in Fig. 2, the input signal is a smooth curve with random noise, and we reconstruct the curve with sparse coding formulation Eq. (2) with DCT as the dictionary. As shown in the second row, we can use few dictionary elements to approximate the signal with the help of ℓ_1 norm on the coefficient vector $\mathbf{x}$. Sparse coding has been applied on many kinds of problems [2] as face recognition [31], image super-resolution [32], image classification [33].

Dictionary learning. As discussed above, the problem of sparse coding focuses on the searching of sparse coefficient vector $\mathbf{x}$. And popular basis functions or vectors are