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Discrete bending forces and their Jacobians ☆

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ABSTRACT

Computation of bending forces on triangle meshes is required for numerous simulation and geometry processing applications. In particular it is a key component in cloth simulation. A common quantity in many bending models is the hinge angle between two adjacent triangles. This angle is straightforward to compute, and its gradient with respect to vertex positions (required for the forces) is easily found in the literature. However, the Hessian of the bend angle, which is required to compute the associated force Jacobians is not documented in the literature. Force Jacobians are required for efficient numerics (e.g., implicit time stepping, Newton-based energy minimization) and are thus highly desirable. Readily available computations of the force Jacobian, such as those produced by symbolic algebra systems, or by autodifferentiation codes, are expensive to compute and therefore less useful. We present compact, easily reproducible, closed form expressions for the Hessian of the bend angle. Compared to automatic differentiation, we measure up to 7× speedup for the evaluation of the bending forces and their Jacobians.

1. Introduction

Important problems in computer simulation, animation, and geometry processing, involve the formulation of an energy in terms of the *hinge angle* between pairs of adjacent mesh triangles (see Fig. 1). Examples include the:

- Wrinkling energy of a worn garment [3,8].
- Elastic energy of a Kirchhoff–Love thin-shell [12].
- Deformation energy for example-driven deformations [9].
- Willmore energy used in mesh smoothing [17].
- Dissipative potential of viscous liquid sheets [4].

Efficient numerical treatments of the associated variational problems (e.g., via Newton's method) or partial differential equations (e.g., via implicit time stepping [3]) necessitate a formulation not only of the energy and its gradient, but also of the *Hessian* of the energy with respect to mesh position.

In our own experience, and over years of interacting with researchers and practitioners working on myriad applications, we have found that these Hessians are exceedingly tedious to derive by hand, with compact formulations sometimes consuming weeks of manual derivation. This process is error prone, often leading to analytic expressions that disagree with numerical validation. The process can be suboptimal, missing opportunities for gathering like terms, thus leading to longer source code and more expensive computation. These liabilities are detrimental to the adoption of efficient numerical methods for hinge-based energies, as evidenced in the literature:

- Bridson et al. [8] avoided Hessians by treating bending forces explicitly; similarly Fröhlich and Botsch[9] avoided Hessians by using Gauss–Newton's method.
- Baraff and Witkin [3] introduced approximating assumptions (e.g, inextensible cloth, undergoing only small deformations, with flat rest shape) treating normals and edge lengths as constants.

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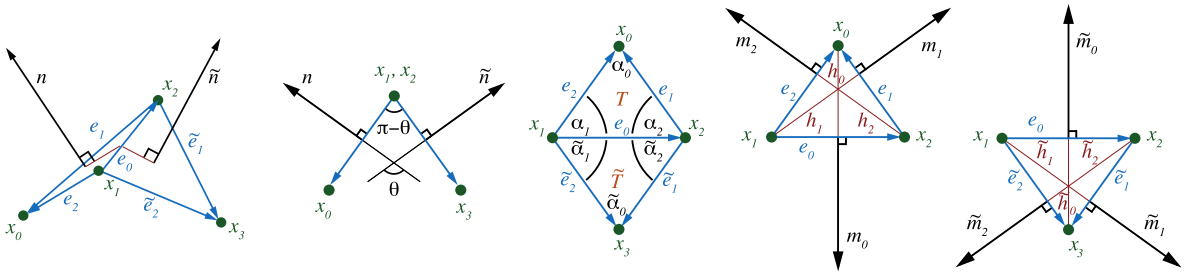

Fig. 1. Vertices, edges, normals and angles around the edge shared by two triangles. The two rightmost schematics show the in-plane edge normals and the associated altitudes from one edge to the opposing vertex. All of these are straightforward to compute given the edge vectors.

- Bergou et al. [6] and, Wardetzky et al. [17] derived a simplified Hessian formula for the special case energy $\sin^2(\theta/2)$, using a technique that does not accommodate the general case.
- Grinspun et al. [12] computed the Hessian using automatic differentiation, which dominated the computational cost of the method.

Contributions. In light of these observations, this paper seeks to facilitate adoption, code legibility, and computational efficiency of hinge-based bending energies.

- We present a compact and efficient formulation of the Hessian for the general case of a hinge-based bending energy.
- By taking advantage of several symmetries in the expressions (some less obvious than others), we observe that many terms can be reused when assembling the Hessian for an entire mesh, further reducing the cost of computation.
- We present the results of experiments documenting up to 7× speedup of the formulation compared to autodifferentiation and up to 4× speedup compared to an existing (but unpublished) symbolic derivation.

2. Bending energy

Notation. Fig. 1 presents the labels and indices for a single *hinge stencil*, consisting of four vertices $\mathbf{x}_i$, five edges $\mathbf{e}_i$ and $\tilde{\mathbf{e}}_i$, two normals $\mathbf{n}$ and $\tilde{\mathbf{n}}$, bend angle θ, interior angles α_i and $\tilde{\alpha}_i$, and heights h_i and $\tilde{h}_i$. Typically, the index i takes on values 0, 1 and 2. Arithmetic on all indices is performed modulo 3. Observe that edges (and all related quantities) are generally labeled the same as the opposing vertices. The tilde decoration is used to distinguish corresponding quantities on the upper and lower triangles T and $\tilde{T}$, respectively. Throughout we use bold letters for vectors. Triangle and edge normals are all assumed to be normalized.

Energy. For a given triangle mesh, consider an arbitrary energy given by a summation over all the interior edges (indexed by i), or "hinges," of a triangle mesh,

$$E(\mathbf{x}) = \sum_i \psi_i(\theta_i), \tag{1}$$

The "bend angle" θ is the angle between the normals of the two triangles incident to the hinge, and $\psi : \mathbb{R} \to \mathbb{R}$ is an application-specific transformation of the bend angle. Drawing from the literature, examples for $\psi_i(\theta_i)$ include

$$\begin{aligned} a_i(\theta_i - b_i)^2 &\quad \textit{Discrete shells } [12]. \\ a_i(\sin(\theta_i/2))^2 &\quad \textit{Discrete Willmore energy } [17]. \\ a_i(\cos(\theta_i/2) - b_i\theta_i) &\quad \textit{Simulation of clothing } [8]. \end{aligned}$$

where a_i and b_i are application-specific scalar coefficients. These typically depend on the local geometry of the mesh and, in physical simulations, the material constitutive properties.

Ref. [8] presented a force, not an energy; above we have integrated the (conservative) force to obtain the corresponding energy. By focusing on the conservative setting, we can roughly halve the computation time, since the conservative force Jacobian is the negated energy Hessian, which is symmetric by definition.

Bending forces and Hessians. We differentiate the energy (1) with respect to vertex positions $\mathbf{x}$ to obtain the bending forces and the energy Hessian

$$\mathbf{f}(\mathbf{x}) = -\sum_i \nabla\psi_i \quad \text{and} \quad H(\mathbf{x}) = \sum_i \text{Hess}(\psi_i).$$

For one particular hinge i, dropping implied subscript from ψ_i and θ_i, the chain rule gives

$$\nabla\psi = \psi'\nabla\theta, \tag{2}$$

$$\text{Hess}(\psi) = \psi'\text{Hess}(\theta) + \psi''\nabla\theta^T\nabla\theta, \tag{3}$$

using the prime to differentiate a univariate function with respect to its scalar argument, e.g., $\psi' = \mathrm{d}\psi/\mathrm{d}\theta$.

Observe that the Hessian of the energy is a weighted sum of Hess (θ) and the outer product $\nabla\theta^T\nabla\theta$ with the same weighting function ψ' appearing in both $\nabla\psi$ and Hess (ψ).

3. Hinge-angle gradient and Hessian

The expression for $\nabla\theta$ has been previously documented in the literature in several forms equivalent to

$$\begin{aligned} \nabla_{\mathbf{x}_1}\theta &= \frac{\cos\alpha_2}{h_1}\mathbf{n}^T + \frac{\cos\tilde{\alpha}_2}{\tilde{h}_1}\tilde{\mathbf{n}}^T, \quad \nabla_{\mathbf{x}_0}\theta = -\frac{1}{h_0}\mathbf{n}^T, \\ \nabla_{\mathbf{x}_2}\theta &= \frac{\cos\alpha_1}{h_2}\mathbf{n}^T + \frac{\cos\tilde{\alpha}_1}{\tilde{h}_2}\tilde{\mathbf{n}}^T, \quad \nabla_{\mathbf{x}_3}\theta = -\frac{1}{\tilde{h}_0}\tilde{\mathbf{n}}^T. \end{aligned} \tag{4}$$

By contrast, the expressions for the hinge angle Hessian are not (to our knowledge) recorded in the literature. Like others, we found the derivation to be extended and