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Dimension-independent simplification and refinement of Morse complexes

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ABSTRACT

Ascending and descending Morse complexes, determined by a scalar field f defined over a manifold M, induce a subdivision of M into regions associated with critical points of f, and compactly represent the topology of M. We define two simplification operators on Morse complexes, which work in arbitrary dimensions, and we define their inverse refinement operators. We describe how simplification and refinement operators affect Morse complexes on M, and we show that these operators form a complete set of atomic operators to create and update Morse complexes on M. Thus, any operator that modifies Morse complexes on M can be expressed as a suitable sequence of the atomic simplification and refinement operators we have defined. The simplification and refinement operators also provide a suitable basis for the construction of a multi-resolution representation of Morse complexes.

1. Introduction

Representing topological information extracted from discrete scalar fields becomes more challenging as more applications, such as analysis and visualization of terrain models, shape and volume data sets, and time-varying volume data sets see increasing amounts of data. Morse theory offers a natural and intuitive way of analyzing the structure of a scalar field as well as of compactly representing the scalar field through a decomposition of its domain into meaningful regions associated with the critical points of the field.

A discrete scalar field f is defined by its values at a finite set V of points on a manifold M in $\mathbb{R}^n$. The discretization of M is often obtained through a simplicial mesh (such as a triangle, or a tetrahedral mesh), or through a regular grid formed by square cells in 2D, or by hexahedral cells in 3D. This geometry-based description provides an accurate representation of a scalar field f, but it fails in capturing compactly its topological structure, defined by critical points and integral lines of f. Beside being compact, a topological description supports also a knowledge-based approach to analyze, visualize and understand the scalar field behavior (in space and time), as required, for instance, in visual data mining applications.

The descending Morse complex is composed of cells defined by the integral lines of f with the same destination. Dually, the ascending Morse complex is composed of cells defined by the integral lines with the same origin. The Morse–Smale complex describes the subdivision of M into cells determined by integral lines with the same origin and destination [40]. These subdivisions have been recognized as convenient representations for analyzing the topology of M, and the behavior of f over M.

Structural problems in Morse and Morse–Smale complexes, like over-segmentation in the presence of noise, or efficiency issues arising because of the very large size of the input data sets, can be faced and solved by defining simplification operators on those complexes and on their topological representations. Morse and Morse–Smale complexes can be simplified by *cancelling* critical points in pairs [35]. Cancellation eliminates two critical points of f, two cells in the Morse complexes, and two vertices in the Morse–Smale complexes. Surprisingly, a cancellation may

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increase the incidence relation on the Morse complexes, and the number of cells in the Morse–Smale ones.

We define two dual dimension-independent simplification operators, which do not have these undesirable properties. The two simplification operators are defined directly on the Morse complexes. We have introduced these operators in [13], where we have shown their effect on a graph-based representation of the two Morse complexes. Here, we describe the effect of these operators on the Morse complexes in detail. We define the inverse refinement operators to the simplification operators, and we describe in detail their effect on the Morse complexes. We show that simplification and refinements operators are valid, i.e., that the result of an application of a feasible simplification or refinement operator on a Morse complex is a Morse complex.

The combination of the simplification and refinement operators defines a minimally complete set of operators for creating and updating Morse decompositions. We prove this result formally by interpreting these operators as Euler operators, i.e., as operators, which affect a constant number of cells in the Euler–Poincaré formula on the ascending and descending Morse complexes. As a consequence, any operator for creating and modifying Morse complexes can be expressed as a suitable sequence of our simplification and refinement operators. In particular, we consider a macro-operator defined in [24] that consists of a 1-saddle-2-saddle cancellation followed by cancellations involving extrema, and we show that this macro-operator can be easily expressed as a sequence of a subset of our operators. The operators we define can be used to generate a multi-resolution model for Morse complexes. The basic ingredients of such model are refinement operators and a suitable dependency relation defined on them.

In summary, contributions of this work include

- a set of simplification operators on the Morse complexes, which:
  - reduce the incidence relation on the Morse complexes, and the number of cells in the Morse–Smale complexes,
  - can be seen as merging or collapsing of cells in the Morse complexes, and
- a set of inverse refinement operators on the Morse complexes.

The simplification and refinement operators

- are defined in arbitrary dimension,
- maintain the topological validity condition expressed by the Euler–Poincaré formula,
- form a minimally complete set of operators for creating and updating Morse complexes, so that any macro-operator can be expressed as a suitable combination of operators in this set, and
- form a basis for a new general multi-resolution model of the Morse complexes.

The remainder of the paper is organized as follows. In Section 2, we review some basic notions on cell complexes and on Morse theory. In Section 3, we discuss related work. In Section 4, we investigate cancellation of critical points of a Morse function f, and its effect on the related Morse and Morse–Smale complexes in arbitrary dimensions. In Section 5, we define two dual simplification operators that we call *removal* and *contraction*, and we describe how ascending and descending Morse complexes are affected by these operators. In Section 6, we define the inverse refinement operators, and we describe how these operators affect the two dual Morse complexes. In Section 7, we give a proof of the validity of the simplification and refinement operators, and we show that they form a basis for the set of operators that modify Morse complexes in a topologically consistent manner. In Section 8, we explain the relationship between cancellation and removal and contraction operators in 3D, and we show how a 1-saddle-2-saddle macro-operator can be expressed through our operators. In Section 9, we draw some concluding remarks and we briefly discuss a multi-resolution model based on the simplification and refinement operators we introduced.

2. Background notions

In this Section, we briefly review some basic notions on cell complexes (for more details on algebraic topology, see [34]). A survey on topological shape representations based on cell complexes is given in [15]. We then review the basic notions of Morse theory in the case of n-manifolds (for more details, see [35,36]).

2.1. Cell complexes

Intuitively, a cell complex is a collection of basic elements, called *cells*, which cover a domain in Euclidean space $\mathbb{R}^m$. A 0-cell is a point in $\mathbb{R}^m$. The boundary of a 0-cell is empty. A *k-dimensional cell* (k-cell) γ in Euclidean space $\mathbb{R}^m$, $1 \leqslant k \leqslant m$, is a subset of $\mathbb{R}^m$ homeomorphic to an open k-dimensional ball $B^k = \{x \in \mathbb{R}^k : \|x\| < 1\}$, with non-null (relative) boundary with respect to the topology induced by the usual topology of $\mathbb{R}^m$ ($\|x\|$ denotes magnitude or norm of a vector x). The integer k is called the *dimension* of a k-cell γ.

An n-dimensional *cell complex* in $\mathbb{R}^m$ is a finite set of cells Γ in $\mathbb{R}^m$ of dimension at most n, $0 \leqslant n \leqslant m$, such that

1. the cells in Γ are pairwise disjoint,
2. for each cell $\gamma \in \Gamma$, the boundary of γ is a disjoint union of cells of Γ.

The boundary of each cell γ in a cell complex Γ is composed of cells of lower dimensions belonging to Γ. The set of all these cells is called the *(combinatorial) boundary* of γ. The *(combinatorial) co-boundary* of γ consists of all cells of Γ that have γ in their combinatorial boundary. If γ is a k-cell, then the *immediate boundary* of γ consists of all $(k-1)$-cells on the boundary of γ, $1 \leqslant k \leqslant n$, and the *immediate co-boundary* of γ consists of all $(k+1)$-cells in the co-boundary of γ, $0 \leqslant k \leqslant n-1$. An h-cell γ' on the boundary of a k-cell γ, $0 \leqslant h \leqslant k$, is called an *h-face* of γ, and γ is called a *coface* of γ'. Each cell γ is a face of itself. If $\gamma' \neq \gamma$, then γ' is called a *proper face* of γ, and γ and γ' are said to be *incident*.