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Accurate Minkowski sum approximation of polyhedral models

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Abstract

We present an algorithm to approximate the 3D Minkowski sum of polyhedral objects. Our algorithm decomposes the polyhedral objects into convex pieces, generates pairwise convex Minkowski sums, and computes their union. We approximate the union by generating a voxel grid, computing signed distance on the grid points, and performing isosurface extraction from the distance field. The accuracy of the algorithm is mainly governed by the resolution of the underlying volumetric grid. Insufficient resolution can result in unwanted handles or disconnected components in the approximation. We use an adaptive subdivision algorithm that overcomes these problems by generating a volumetric grid at an appropriate resolution. We guarantee that our approximation has the same topology as the exact Minkowski sum. We also provide a two-sided Hausdorff distance bound on the approximation. Our algorithm is relatively simple to implement and works well on complex models. We have used it for exact 3D translation motion planning, offset computation, mathematical morphological operations, and bounded-error penetration depth estimation.

Keywords: Minkowski sum; Configuration space; Offset; Computer-aided design

1. Introduction

The problem of Minkowski sum computation arises in many applications including solid modeling, digital geometry processing, robotics, dynamic simulation, and computer animation. The Minkowski sum of two sets P and Q is the set of points $\{\boldsymbol{p}+\boldsymbol{q}|\boldsymbol{p}\in P, \boldsymbol{q}\in Q\}$. Minkowski sum has a number of applications. They are useful as a tool to compute collision-free paths in robot motion planning [20], computer-aided design and manufacturing [18], satellite layout [4], penetration depth computation and dynamic simulation [16]. They have also been used for morphing [15], offset computation [21], and mathematical morphological operations [23].

Our goal is to compute the boundary of the 3D Minkowski sum of two polyhedral models. The Minkowski sum of two convex polytopes (with n features) can have $O(n^2)$ combinatorial complexity and is relatively simple to compute. On the other hand, the Minkowski sum of non-convex polyhedra can have complexity as high as $O(n^6)$ [13]. One of the commonly used approach to compute Minkowski sums decomposes the two non-convex polyhedra into convex pieces, computes their pairwise Minkowski sums, and finally the union of the pairwise Minkowski sums. The main bottleneck in

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implementing such an algorithm is computing the union of pairwise Minkowski sums. Given m polyhedral primitives, their union can have combinatorial complexity $O(m^3)$ [2] and m can be high in the context of Minkowski sum computation (e.g., a few thousand). Furthermore, robust computation of the boundary of the union and handling all degeneracies remains a major issue [13,1]. As a result, no practical algorithms are known for robust computation of exact Minkowski sum of complex polyhedral models.

1.1. Main results

We present a novel algorithm to approximate the Minkowski sum of polyhedral models. Instead of computing the exact union, we use distance field-based techniques to approximate the union of the pairwise Minkowski sums. Our algorithm generates an adaptive volumetric grid, computes a distance field, and performs isosurface extraction from it to obtain an approximation to the Minkowski sum. The accuracy of the algorithm is mainly governed by the rate of sampling, i.e., the resolution of the underlying volumetric grid. Insufficient resolution can result in unwanted handles or disconnected components in the approximation. Due to lack of resolution, the approximation may not capture many of the features, e.g., small holes, present in the exact Minkowski sum. We use an adaptive subdivision algorithm that generates a volumetric grid at a sufficient resolution such that a faithful approximation can be obtained by performing isosurface extraction on the resulting grid. We ensure a good quality of approximation by guaranteeing the correct topology as well as bounding the two-sided Hausdorff distance between the approximation and the exact Minkowski sum.

In order to speed up the computation, we employ two types of culling techniques during adaptive subdivision. Our algorithm performs *cell culling* to eliminate the grid cells that do not contain a part of the Minkowski sum boundary. Our algorithm also takes advantage of *primitive culling* and performs efficient distance and inside/outside queries by only considering a small subset of primitives, while preserving the correctness of these queries. In practice, these culling techniques improve the performance of the algorithm by more than *two orders* of magnitude.

We have used our Minkowski sum approximation algorithm for a number of applications. These include:

- Exact robot motion planning of robots with translation degrees of freedom.
- Offsets and mathematical morphological operations.
- Penetration depth estimation between overlapping polyhedra with tight error bounds.

Our algorithm is simple to implement and we have tested its performance on a number of benchmarks. The underlying polyhedral models consist of several hundreds of triangles. The computation of Minkowski sum takes few minutes on a 2 GHz Pentium IV processor.

Some of the novel results of our approach include:

- Approximate algorithm for computing Minkowski sum of polyhedral models.
- Culling techniques to improve the performance of adaptive subdivision and sampling scheme.
- Guaranteed topology and two-sided Hausdorff distance bounds on the approximation.
- Application to motion planning, offset, and penetration depth computation.

To the best of our knowledge, this is the first algorithm that can compute a topologically accurate approximation of Minkowski sum of complex polyhedral models.

1.2. Organization

The rest of our paper is organized as follows. In Section 2, we review the earlier work on Minkowski sum computation. Section 3 gives an overview of our approach. In Section 4, we present our approximate algorithm to compute the boundary of Minkowski sum. Sections 5 discusses its application to motion planning, offsets and morphological operations, and penetration depth computation. We highlight its performance on various benchmarks in Section 6. Sections 7 discusses some limitations of our approach.

2. Previous work

In this section, we give a brief survey of the related work. Many algorithms have been proposed for Minkowski sum computation in computational geometry and solid modeling [20,12,9,11]. A survey can be found in [13].